LOW-DIMENSIONAL SPECTRAL TRUNCATIONS FOR TAYLOR-COUETTE FLOW

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Abstract

Experiments have shown that low–dimensional attractors exist in chaotic fluid flows, implying that only a small number of degrees of freedom are active. Current spectral computational techniques demand that the solution be represented as a sum of separable functions of space and time; hence, many more modes than the dimensionality of the attractor are needed to resolve the solution. The question we address in this paper is whether one can exploit the fact that the dynamics is low dimensional to substantially reduce the number of computational modes, without sacrificing quantitative accuracy. To do so would be of great practical importance. Using the formalism of inertial manifold theory, we discuss a novel procedure for constructing a low–dimensional truncation of the Navier–Stokes equations with no adjustable parameters, in which an initially small set of basis functions is augmented as needed during time integration. The method will be applied to temporally quasiperiodic and chaotic Taylor–Couette flows, with the choice of basis functions based on previous numerical work. Implementation is in progress.

1 INTRODUCTION

Recently, there has been a rapid advance in the study of strongly nonlinear systems, primarily through the use of numerical simulation. However, it has become clear that many interesting problems in which a wide range of length and time scales are active will not be computationally accessible in the foreseeable future, despite improvements in computer hardware. To make further progress, methods of solving partial differential equations (PDE's) will be needed that are substantially more efficient than those currently in use. For finite difference computations, the use of adaptive grid solvers[1] (which vary the grid resolution over the domain in order to focus computational work on the most complex regions), has been an important innovation. In this paper, we discuss in very preliminary terms the development of an adaptive spectral method, *i.e.* a spectral, initial–value algorithm in which an initial, optimal set of basis functions is chosen, with more basis functions added as required during the calculation.

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Traditional spectral methods represent the solution to a PDE as a sum of known basis functions (such as sines and cosines) multiplied by time-dependent amplitudes; the latter are the computational variables[2]. They are most useful in problems where the geometry of the domain possesses a high degree of symmetry; most often in closed flows. Frequently, these problems also have interesting regimes which are described by relatively low-dimensional attractors, which suggests that there may exist a preferred description in which the number of computational variables N needed is comparably low. Assuming that prior analysis of the flow has led to a small set of physically well-motivated basis functions (which are adequate in a limited regime of parameter space — similar to an amplitude expansion), we can augment the set by generating additional basis functions from the nonlinear terms in the Navier–Stokes equation. This is necessary because nonlinear interactions generically introduce directions in phase space not spanned by the original set, and in strongly nonlinear problems these will be non-negligible. Because the fundamental low dimensionality of the attractor is not affected by nonlinearity, this procedure should converge, but it is important to note that it is *not* true that a PDE with an attractor of low dimension D will generally be well represented using $N \sim D$ variables.

As an example, consider traveling waves, which are space-time periodic (in x say), and thus have a one dimensional attractor. They can be generically described as a spectral sum (assuming an infinite or periodic domain)

$$V(\mathbf{r},t) = \sum_{m=-\infty}^{\infty} b_m e^{im(x-ct)} \tag{1}$$

where c is the phase speed of the wave. The coefficients b_m (which may be functions of the other spatial coordinates) are non-vanishing for an infinite number of m's, and no coordinate transformation can change this fact. The system point lives on a one dimensional curve in phase space where there are no constraints on the representation of the solution. However, spectral numerical methods *always* demand that the solution be represented as a sum of the form

$$V(\mathbf{r},t) = \sum_{m=-\infty}^{\infty} f_m(t) g_m(\mathbf{r}) \tag{2}$$

where the g_m are members of a complete set, and may themselves be written as sums of simple functions. When separability is imposed on the computational variables we find we need $N \gg D$ for adequate resolution. This is so regardless of whether the attractor is smooth or 'strange'.

On the other hand, it is intuitively clear that for most purposes an infinite number of coefficients is unnecessary; for a smooth solution of the form in equation 1 the magnitudes of the coefficients decrease exponentially for large m, and a finite number of them will be enough. Similarly, given that all temporally organized flows and many weakly turbulent flows possess a high degree of spatial order, it is reasonable to hypothesize that, with an astute choice of basis functions, the solutions should be well represented with N of order $10D$ to $100D$. The problem then reduces to computing the 'right' basis functions, a highly nontrivial task. In this we rely on a good understanding of the physics of the problem, through full numerical simulations at a few parameter values.

Steady or time periodic flows, no matter how spatially complex, have linear eigenmodes which are temporally simple. The spatial structure of these eigenmodes is a good representation of the the physical instabilities in the flow, they can also be computed in a straightforward manner; hence, they are a sensible choice for the starting components of a spectral basis set. We adapt the set of basis functions as follows: In the usual spectral representation, a complete set of known basis function are truncated to a finite, incomplete set. The nonlinear terms in the PDE generate components which lie outside the space spanned by the truncated set. These are ignored, but a quantitative estimate can be made of the error incurred in doing so. If the error becomes too large, the calculation can easily be refined by including more basis functions. Suppose, however, that one has a finite set of functions which are computed in some arbitrary manner so that they are not an obvious subset of a complete and infinite set of basis functions. If, by some quantitative diagnostic these are determined to be insufficient for accurate computations, then a method must be found of generating additional functions to augment the basis. We outline such a method below, drawing on the conceptual structure of inertial manifold theory as developed by Jolly *et. al.* , Titi, and Foias *et. al.* [3,9,10]

The rest of the paper is organized as follows. In section 2 we review the Taylor–Couette flows that are relevant to our discussion, and describe the physical problem that will be used as a test case for our method. In section 3 we present a brief overview of inertial manifold theory, and describe our algorithm, which we refer to as an adaptive spectral method. Section 4 contains our discussion and conclusions.

2 TAYLOR–COUETTE FLOW

We consider the set of states found in narrow gap systems where the outer cylinder is held fixed and the inner cylinder speed increased. The bifurcation parameter is the Reynolds number $R \equiv \Omega a(b-a)/\nu$, where Ω is the inner cylinder frequency, ν is the kinematic viscosity of the working fluid, and a and b are the inner and outer cylinder radii respectively. We nondimensionalize setting the gap width, the inner cylinder speed and the fluid density equal to one. In our computations $a/b = .875$ and we impose strict periodicity in the axial and azimuthal directions with axial wavelength λ and azimuthal wavenumber s.

At $R \equiv R_c$ there is a bifurcation to axially periodic, axisymmetric Taylor vortex flow (TVF), consisting of toroidal vortices wrapped around the inner cylinder. At R a few percent above R_c, a supercritical Hopf bifurcation leads to wavy vortex flow (WVF), in which azimuthally traveling waves appear on the vortices. These are characterized by azimuthal wavenumber m_1 and phase speed c_1. At still higher R the periodic flow becomes modulated (modulated wavy vortex flow or MWV), with the power spectrum characterized by the presence of two incommensurate frequencies. The flow is quasiperiodic relative to the laboratory frame, but periodic in the frame rotating with the original traveling wave. The azimuthal periodicity of the flow may change in this transition. In experimental work Gollub & Swinney[4] found, as R is further increased, a non-hysteretic transition from MWV to chaos. Further experimental work[5] showed that a low–dimensional attractor exists for the chaotic flow. Because of the rotational symmetry MWV is a limit cycle in the correct frame; hence, the transition directly to chaos is not understood by current theories.

In previous work[6], we solved the Navier–Stokes equations using a pseudo-spectral initial value code, and showed that several distinct branches of modulated waves exist, and that several routes to chaos are exhibited by this system. Relevant to this paper is a numerically discovered mode competition between two simultaneously growing Floquet modes (linear eigenmodes of the unstable WVF), which we have termed 'ZS' and 'GS' modes. Our observations have been experimentally confirmed[7].

2.1 Instability of WVF

We have discussed elsewhere the role of the outflow jet between adjacent Taylor vortices in this sequence of instabilities. The radial outflow produces a correspondingly strong azimuthal jet, which becomes sharper and narrower as R increases, for both TVF and WVF. The WVF becomes unstable to a Floquet mode, which has space-time dependence of the form $e^{i(m_2\theta-\omega t)}\mathbf{f}(r,\theta,z,t)$. Here m_2 is the azimuthal wavenumber of the modulation, ω is real at onset, and $\mathbf{f}$ has the space-time periodicity of the unstable WVF. This functional form is consistent with the patterns observed by Gorman & Swinney, and the theoretical work of Rand[8].

In the frame rotating with the initial WVF, the MWV is time periodic and it is trivial to separate the steady–state flow (defined as $\bar{\mathbf{v}}$) from the fluctuating (or quasiperiodic) piece ($\mathbf{v}^{qp}$). The latter is equal to the Floquet mode near onset. Spatially, the Floquet mode is concentrated in the region of the outflow jet and consists of a set of small vortices, which are best visualized by plotting the azimuthal vorticity ω_θ^{qp} , as in figure 1. Contours of ω_θ^{qp} are plotted on an unrolled cylindrical surface parallel to the inner cylinder at mid-gap for the ZS (1a) and GS (1b) modes. In both cases, the fluctuating field consists of compact, three-dimensional vortex pairs sitting exactly on top of the outflow. For the ZS mode in figure 1a, $m_2 = 3m_1$, and for the GS flow in figure 1b $m_2 = m_1$. As a function of time, the vortex pairs drift along the outflow at a mean speed equal to the modulation frequency (relative to the frame rotating with the underlying WVF) divided by the wavelength $2\pi/m_2$ of the Floquet mode. The visual signature of the ZS mode is small, rapidly moving ripples at the outflow, and of the GS mode a periodic flattening of the outflow contour. The space time symmetry of these two modes is identical, thus mode competition can occur.

As a function of R, we see the following sequence of flows[7] (with $s = 4$ and $\lambda = 2.5$): At $R/R_c = 8.50$ there is a ZS equilibrium with the GS mode evident as a decaying transient. At $R/R_c = 9.80$ the GS mode is stable, and transients are dominated by the decaying ZS mode. There is an intermediate range of R near $9R_c$ where both modes are present with positive growth rate, and the flow appears to be chaotic. The long term temporal behavior of the flow over this range is as yet undetermined. We infer that the growth rate of the ZS mode is positive below about $R/R_c = 9.0$ and negative above, while the growth rate of the GS mode is negative below roughly $R/R_c = 9.5$ and positive above. Presumably, in an extended parameter space there is a codimension two point at which both their eigenvalues cross the imaginary axis simultaneously. To find such a point through the computations is prohibitively expensive; hence this information is not useful here.

3 LOW–DIMENSIONAL MODEL

We are interested in the physics and mathematics of this problem in itself: if, when and how the mode competition leads to chaos, whether there is mode locking, what the effects of the rotational symmetry are on the dynamics, *etc.* Fortunately, it is also an ideal test problem for the development of adaptive spectral methods: it is a natural candidate for the spectral decomposition described above, and we can make precise quantitative comparison between the new method and both experiments and traditional spectral computations. In addition, the behavior of the system is not known *a priori* so we can be sure that we are not building the results into our model. Before describing the algorithm, we briefly outline some ideas from inertial manifold theory which are used in our discussion[9].

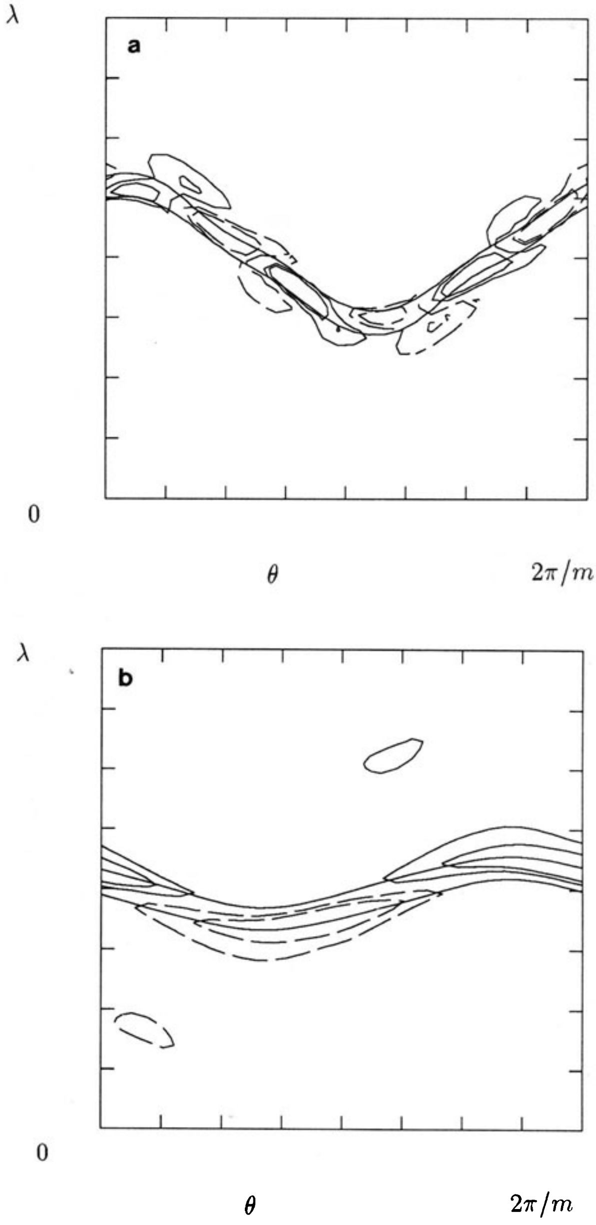

Figure 1. Contours of ω_θ^{qp} in the (θ, z) plane at midgap for a) the ZS mode at $R/R_c = 8.5$, $m_2 = 3m_1$, contour interval $= \pm 0.1$, and b) the GS mode at $R/R_c = 9.8$, $m_2 = m_1$,contour interval $= \pm 0.3$. Dashed contours represent negative values, solid contours are positive. The outflow jet has been indicated by the solid contour on which $v_\theta = 0.2$.

3.1 Inertial manifold theory

An inertial manifold $\mathcal{M}$ for a PDE, if it exists, is a a smooth, finite dimensional manifold in the (infinite dimensional) phase space of the system which attracts trajectories exponentially. When $\mathcal{M}$ exists, asymptotically the entire dynamics of the PDE is described by a finite-dimensional system of ordinary differential equations (ODE's), called the inertial form, which is obtained by restricting the equations of motion to $\mathcal{M}$. If a global attractor $\mathcal{A}$ exists then $\mathcal{A} \subset \mathcal{M}$. In general $\mathcal{A}$ may not be smooth, while $\mathcal{M}$ is smooth by definition, and $\mathcal{A}$ may attract trajectories arbitrarily slowly. Thus $\mathcal{M}$ is the more practical object to work with. The goal of the theory is to construct an approximate inertial manifold (AIM) explicitly, project the equations onto this manifold, and solve the simpler system. Existing work suggests that such a procedure can lead to significant improvements in the truncation error[10]. Unlike center manifold equations, the inertial form may be valid for many bifurcations, as long as the global attractor remains in the space defined by the AIM.

Formally, one separates the PDE into a linear and a nonlinear term (which we assume without loss of generality to be quadratic):

$$\dot{u} = \mathcal{L}u + \mathcal{F}(u, u). \tag{3}$$

We define $\mathcal{P}$ as the projection onto a finite subspace of the full phase space $\mathcal{H}$, and $\mathcal{Q} \equiv \mathcal{I} - \mathcal{P}$ (where $\mathcal{I}$ is the identity operator) as the complementary projection. Setting $p = \mathcal{P}u$ and $q = \mathcal{Q}u$ we obtain:

$$\dot{p} = \mathcal{PL}p + \mathcal{PF}(p+q, p+q) \tag{4}$$

$$\dot{q} = \mathcal{QL}q + \mathcal{QF}(p+q, p+q). \tag{5}$$

Note that if the basis functions p, q are chosen to be eigenfunctions of $\mathcal{L}$, then the projection operators can be dropped from the linear terms. If $\mathcal{M}$ is of dimension less than or equal to the dimension of $\mathcal{PH}$, then formally it can be defined as the graph[9] of a function $q = \Psi(p)$. Thus the p's are the coordinates on $\mathcal{M}$ and the q's are the coordinates over the rest of the space. Clearly, how the space is to be divided into p's and q's is an important consideration, which is in general problem dependent.

To construct the inertial form one must calculate the function $\Psi(p)$ which models the action of the neglected modes, through the nonlinearity, on the modes that are kept. To calculate it exactly requires solving the full problem. Several methods of constructing an approximating function $\bar{\Psi}(p)$ have been proposed[9], which rely on iterating some form of equation 5 with $\dot{q}$ set equal to zero, and are thus defined satisfactorily only for steady states. An approximate function $q = \bar{\Psi}(p)$ defines the AIM, and the inertial form is constructed by replacing equation 4 with

$$\dot{p} = \mathcal{PL}p + \mathcal{PF}(p + \bar{\Psi}(p), p + \bar{\Psi}(p)) \tag{6}$$

and dropping equation 5. For example, in a traditional spectral method the solution u is represented as a truncated sum of N basis functions. These are the p's, the neglected basis function with index greater than N are the q's, and $\mathcal{M}$ is approximated by $q = \bar{\Psi}(p) = 0$.

3.2 Adaptive spectral methods

In this section we discuss our proposed method in the context of the Taylor–Couette problem described above. We use the formal language of inertial manifold theory, but forgo any attempts to be rigorous.

Let $\mathbf{u}_0$ be the equilibrium WVF solution to the Navier–Stokes equations in the appropriate frame (so it is steady state). For an arbitrary solution $\mathbf{u}$ we set $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}'$. We define the operator $\mathcal{L}(\mathbf{u}_0)$ to be the Navier–Stokes linearized around $\mathbf{u}_0$, and the bilinear operator $\mathcal{B}(\mathbf{u}, \mathbf{v}) \equiv (\mathbf{u} \cdot \nabla)\mathbf{v}$. With this definition, $\mathcal{L}(\mathbf{u}_0)$ has two growing eigenmodes, which are the GS and ZS Floquet modes. Let $\mathcal{P}$ be the projection onto the space spanned by these modes (which has dimension 4), $\mathcal{Q}$ be the complementary projection, and set $p = \mathcal{P}\mathbf{u}'$ and $q = \mathcal{Q}\mathbf{u}'$.

We first define our AIM by setting $q = \bar{\Psi}(p) = 0$, and solve

$$\dot{p} = \mathcal{L}(\mathbf{u}_0)p + \mathcal{P}\mathcal{B}(p,p) \tag{7}$$

using an initial value algorithm. We also monitor the size of the neglected term $\mathcal{Q}\mathcal{B}(p,p)$, which is a quantitative measure of how well this AIM approximates the true local (in time) solution. When this term becomes significant, we define a new basis function, orthogonal to the original set but capturing the structure along the escaping direction. We then continue the computation using the augmented basis set, redefining $\mathcal{P}$ and $\mathcal{Q}$ appropriately.

Enlarging the basis set $p \rightarrow p' = p + \phi(p)$ in equation 4 (where $\phi(p)$ is the piece of $\mathcal{B}(p,p)$ that does not lie in span$\{p\}$) is similar to defining a new AIM by $q = \bar{\Psi}(p) = \phi(p)$. Thus we are allowing the initial value computation to locally define the best approximation to $\mathcal{M}$. Also, because we keep the dynamical equation for $\phi(p)$, our method is effectively following the deformations of this AIM as a function of time. When necessary, another basis function will be added, which corresponds to further refinement of the AIM. Note there is no need to assume here a clean separation of the large, slow scales from the short, fast scales.

Beyond the first iteration the situation is complicated somewhat by the fact that the linear operator $\mathcal{L}(\mathbf{u}_0)$ no longer commutes with the projection operators. The construction of these operators, and the matrix of coupling coefficients entailed by $\mathcal{B}$, is computationally intensive, however it needs to be done only once for each round of time integration. The time integration itself is trivial. The fact that the spatial structures present in the physical flow are well represented by the Floquet modes implies that the process should converge rapidly to a 'good enough' set of coordinates (or basis functions) p. This method can also be used to compute the unstable steady state WVF in this parameter regime, and in that case should be exact since the only unstable directions are those defined by the Floquet modes.

4 DISCUSSION

We have presented an algorithm for computing accurate solutions to the Navier–Stokes equations for Taylor–Couette flow, using a very small set of ordinary differential equations, the implementation of which is still in progress. We have described a physical flow, accessible in the laboratory, which will be used as a test case for the numerical procedure. We refer to the procedure as an adaptive spectral method: spectral because we convert the PDE to a set of ODE's by representing it as a sum of spatial functions multipled by time dependent amplitudes, and adaptive because we follow the calculation as it evolves along new directions in phase space and use these to construct new basis functions.

We would like to emphasize that, unlike the approaches developed by Aubrey, Lumley and others[11], we are not trying to derive from the full problem a simpler, approximate model with qualitatively similar dynamics. Our aim in this work is to maintain the high accuracy of full simulation while vastly increasing the efficiency of our computations. We do not claim that our method is rigorously justified; in fact, the mathematical underpinnings of our algorithm could be considered somewhat tenuous. It is thus essential to be able to test results from the truncated system against both full simulations and laboratory experiments. We will then be in a position to state definitively whether this procedure is viable. We hope that this work will lead to generally useful algorithms for studying systems which combine nontrivial spatial structure with complex, but low-dimensional, temporal dynamics.

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