

STELLAR CONVECTION. III. CONVECTION AT LARGE RAYLEIGH NUMBERS

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ABSTRACT

We present the results of a numerical, three-dimensional study of convection in a self-gravitating sphere of Boussinesq fluid with a Rayleigh number of 10^{10} and a Prandtl number of 1. The velocity and temperature are computed by using spectral methods (spherical harmonics, $Y^{l,m}$, $l \leq 12$) in the horizontal and finite-differencing in the radial directions. An eddy viscosity and diffusivity are needed to model the subresolution flow. The amplitudes of the eddy viscosity and diffusivity are determined by requiring that the time-averaged kinetic and thermal energy spectra remain unchanged when the limit of spatial resolution varies. For Rayleigh numbers much less than 10^{10} the flows do not have well-defined inertial ranges, and an eddy viscosity and diffusivity cannot be assigned in this self-consistent manner. By computing the energy spectra as well as the detailed energy budgets as a function of wavenumber, we show that for $Rs = 10^{10}$ there is an inertial range for the modes corresponding to spherical harmonics with $l > 6$. We also show that the velocity field becomes nearly isotropic at small wavelengths. Intermittent bursts of convective flux and energy propagate inward from the outer boundary layer. The bursts are shown to cascade from large to small wavelengths.

Subject headings: convection — hydrodynamics — stars: interiors

I. INTRODUCTION

This is the third in a series of papers on convection in spheres. In it we extend our numerical calculations to high Rayleigh numbers. In the second paper of this series (Marcus 1980a), the Reynolds and Péclet numbers were small enough that the small length scales where dissipation occurs could be resolved. In this paper the dissipative length scales are much smaller than the numerical resolution, so it is necessary to model the small-wavelength end of the kinetic and thermal energy spectra. Clearly, no attempt to compute the convective flows in the cores of stars (where the Reynolds numbers are greater than 10^{10}) is able to resolve all of the scales of motion so it is necessary to model the unresolved part of the flow. For example, the mixing-length treatment of convection and the spectral method proposed by Ledoux, Schwarzschild, and Spiegel (1961) both assume that buoyancy drives the convection in a narrow band of large-wavelength modes. Both methods utilize the model of homogeneous, isotropic turbulence in which buoyancy is not important to compute the effects of the small-scale unresolved motions.

In an actual sphere of convecting fluid we might expect that the buoyancy effects are felt over a wide range of wavelengths, and that the fluid which is strongly driven by the buoyancy in the radial direction is neither uniform nor isotropic for an even wider range of wavelengths. In this paper we explicitly compute the fluid's velocity and temperature at the largest wavelengths and confirm that the flow is anisotropic and that the energy spectra are not smooth functions of wavelength but have a large amount of fine structure. At smaller wavelengths we also compute the flow and show that it becomes isotropic and that the energy spectra develop inertial ranges with the smooth $\sim k^{-5/3}$ power law associated with homogeneous turbulence. We are then able to model the effects of the small-scale fluid motions by assuming that the calculated inertial range smoothly continues beyond our limit of spatial resolution. In § II we show how to parametrize the transport properties of the unresolvable inertial subrange with eddy viscosities and diffusivities. The time-averaged results of our calculations are presented in § III. In § IV we examine the time-dependent fluctuations in the energy spectra and show how they cascade from large to small wavelengths. A summary of our conclusions appears in § V.

II. EDDY VISCOSITY AND DIFFUSIVITY

a) Motivation

Unfortunately, it is impossible to compute numerical solutions to the Navier-Stokes equation for large Reynolds number flows. The Reynolds number in the convective core of an O star is $\sim 10^{12}$, which means that the ratio of the pressure scale height to the dissipative length scale is $\sim 10^9$. Unless a numerical method has the capability of

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resolving this range of scales, the resulting solution will lack some essential physics. To circumvent their inability to resolve the dissipative scales, hydrodynamicists typically model the dissipation by replacing the true kinetic viscosity, ν , with an eddy viscosity, ν_e , where $\nu_e \gg \nu$. It is usually argued (see Deardorff 1972 and references therein) from dimensional analysis that ν_e should be of the form

$$\nu_e = c L_{\text{res}} V_{\text{res}}, \quad (2.1)$$

where L_{res} and V_{res} are the length and velocity at the limit of the spatial resolution and c is a constant of order unity that is independent of both position and L_{res} . Similar reasoning shows that an eddy diffusivity, κ_e , of the same form as equation (2.1) should be included in the heat diffusion equation. More complicated models for eddy viscosities exist (see Kraichnan 1976). However, a simple test of the validity of an eddy viscosity of the form of equation (2.1) is to see if we can find a value of c (independent of position) such that all solutions computed with this value of c , but with different limits of spatial resolution, have the same energy spectra. This type of test is easily performed with a modal analysis.

b) Definitions and Their Implications

Let the temperature T be a sum of its resolvable \hat{T} and unresolvable T' parts. If we assume that the radial resolution is much greater than the angular resolution, then we may write

$$\hat{T} = \sum_{l=0}^{l_{\text{cutoff}}} \sum_{m,\gamma} T_{\gamma,l,m}(r,t) Y^{\gamma,l,m} = \sum_{l=0}^{l_{\text{cutoff}}} T_l \quad (2.2)$$

and

$$T' = \sum_{l=l_{\text{cutoff}}+1}^{\infty} \sum_{m,\gamma} T_{\gamma,l,m}(r,t) Y^{\gamma,l,m} = \sum_{l=l_{\text{cutoff}}+1}^{\infty} T_l. \quad (2.3)$$

The sums in equations (2.2)–(2.3) are over $0 \leq m \leq l$ with γ equal to R and I , where

$$\begin{aligned} Y^{R,l,m} &\equiv 2(2\pi)^{1/2} \text{Re}(Y^{l,m}) & (m \neq 0) \\ &\equiv 2(\pi)^{1/2} Y^{l,0} & (m = 0), \end{aligned} \quad (2.4)$$

$$\begin{aligned} Y^{I,l,m} &\equiv 2(2\pi)^{1/2} \text{Im}(Y^{l,m}) & (m \neq 0) \\ &\equiv 0 & (m = 0), \end{aligned} \quad (2.5)$$

and where $\text{Re}(Y^{l,m})$ and $\text{Im}(Y^{l,m})$ are the real and imaginary parts of the spherical harmonic. We use the notation throughout that any quantity written with subscripts γ, l, m is a function only of radius and time and corresponds to the (γ, l, m) component of a quantity, i.e.,

$$f_{\gamma,l,m}(r,t) \equiv \frac{1}{4\pi} \int_{4\pi} Y^{\gamma,l,m} f(r, \theta, \phi, t) d\Omega. \quad (2.6)$$

Writing $\mathbf{v} = \hat{\mathbf{v}} + \mathbf{v}'$, we can define an eddy diffusivity for each mode, $\kappa_e^{\gamma,l,m}(l_{\text{cutoff}}, r, t)$ such that

$$\nabla \cdot [\kappa_e^{\gamma,l,m} \nabla (\hat{T}_{\gamma,l,m} Y^{\gamma,l,m})] - [(\hat{\mathbf{v}} \cdot \nabla) \hat{T}]_{\gamma,l,m} Y^{\gamma,l,m} \equiv -[(\mathbf{v}' \cdot \nabla) T]_{\gamma,l,m} Y^{\gamma,l,m}. \quad (2.7)$$

All of the quantities on the left-hand side of equation (2.7) are numerically calculable except for $\kappa_e^{\gamma,l,m}$. If \mathbf{v} and T were known over all scales, we could then solve for $\kappa_e^{\gamma,l,m}(l_{\text{cutoff}}, r, t)$ and determine how strongly it depends on γ, l, m, r, t , and l_{cutoff} . Similarly we can define two eddy viscosities for each (γ, l, m) component of the velocity. There are two viscosities because the velocity has two degrees of freedom.

The toroidal eddy viscosity, $\nu_T^{\gamma,l,m}(l_{\text{cutoff}}, r, t)$, is defined as

$$\hat{\mathbf{e}}_r \cdot \{ \nabla \times [(\hat{\mathbf{v}} \cdot \nabla) \hat{\mathbf{v}} - (\mathbf{v}' \cdot \nabla) \mathbf{v}] \}_{\gamma,l,m} = \hat{\mathbf{e}}_r \cdot \nabla \times [\nabla \cdot (\nu_T^{\gamma,l,m} \nabla \hat{\mathbf{v}})]_{\gamma,l,m}; \quad (2.8)$$

and the poloidal eddy viscosity, $\nu_p^{\gamma,l,m}(l_{\text{cutoff}}, r, t)$, is defined as

$$\hat{\mathbf{e}}_r \cdot \{ (\hat{\mathbf{v}} \cdot \nabla) \hat{\mathbf{v}} - (\mathbf{v}' \cdot \nabla) \mathbf{v} - \nabla P^{im} \}_{\gamma,l,m} = \hat{\mathbf{e}}_r \cdot [\nabla \cdot (\nu_p^{\gamma,l,m} \nabla \hat{\mathbf{v}})]_{\gamma,l,m}. \quad (2.9)$$

The pressure, P^{im} , that appears in equation (2.9) is the part of the resolvable pressure that is due to nonlinear terms involving \mathbf{v}' :

$$\nabla^2 (P^{im}_{\gamma,l,m} Y^{\gamma,l,m}) = - \{ \nabla \cdot [(\mathbf{v}' \cdot \nabla) \mathbf{v} - (\hat{\mathbf{v}} \cdot \nabla) \hat{\mathbf{v}}] \}_{\gamma,l,m} Y^{\gamma,l,m}. \quad (2.10)$$

It is necessary to include P^{im} in equation (2.9) to make $\langle (\hat{\mathbf{v}} \cdot \nabla) \hat{\mathbf{v}} - (\mathbf{v}' \cdot \nabla) \mathbf{v} - \nabla P^{im} \rangle$ divergenceless. The resolvable pressure $\hat{P}_{\gamma,l,m}$ is then equal to

$$\hat{P}_{\gamma,l,m} = P^{ex}_{\gamma,l,m} + P^{im}_{\gamma,l,m}, \quad (2.11)$$

where P^{ex} is obtained from the divergence of the Navier-Stokes equation (see Marcus 1980a, eq. [2.12]):

$$\nabla^2(P^{ex}_{\gamma,l,m} Y^{\gamma,l,m}) = -\{\nabla \cdot [(\hat{v} \cdot \nabla)\hat{v}]\}_{\gamma,l,m} Y^{\gamma,l,m} + r^{-2} \text{RsPr} \partial(r^3 \hat{T}_{\gamma,l,m}) / \partial r Y^{\gamma,l,m}.$$

Equations (2.8) and (2.9) provide a complete description for the resolvable part of $[(\hat{v} \cdot \nabla)\hat{v} - (\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla P^{im}]$:

$$\begin{aligned} [(\hat{v} \cdot \nabla)\hat{v} - (\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla P^{im}] = & \sum_{\gamma,l,m} [l(l+1)]^{-1} (\nabla \times [\nabla \times \{r^2 \hat{e}_r Y^{\gamma,l,m} [\hat{e}_r \cdot (\nabla \cdot V_p^{\gamma,l,m} \nabla \hat{V})]_{\gamma,l,m}\}]) \\ & + \nabla \times [r^2 \hat{e}_r Y^{\gamma,l,m} \{\hat{e}_r \cdot [\nabla \times (\nabla \cdot V_T^{\gamma,l,m} \nabla \hat{V})]_{\gamma,l,m}\}]. \end{aligned} \quad (2.13)$$

If $v_T = v_p$, one might conclude from equations (2.8) and (2.9) that $[(\hat{v} \cdot \nabla)\hat{v} - (\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla P^{im}]$ is equal to $\nabla \cdot (v_{\mu} \nabla \hat{v})^{\phi}$ but this is not true because the latter quantity is not divergenceless. When modeling \mathbf{v}' with an eddy viscosity, one must be careful that the model is realizable; that is, $(\mathbf{v}')^2$ should never be negative. This problem is frequently encountered (see Schlesinger 1978) but does not occur in our modal definitions of eddy viscosity.

For the eddy viscosities and diffusivity to be numerically useful tools, they must be independent of γ and m . At large wavenumbers, we expect the temperature and velocity to be isotropic. Furthermore, if the nonlinear terms are dominated by local interactions in phase space (i.e., the interaction of $v_{\gamma',l',m'}$ with $T_{\gamma'',l'',m''}$ is an important contribution to $[(\mathbf{v} \cdot \nabla)T]_{\gamma,l,m}$ only if $l' \approx l'' \approx l$), then we expect $\kappa_e^{\gamma,l,m}$ to be independent of γ and m for large values of l . For small values of l the flow is not isotropic and $\kappa_e^{\gamma,l,m}$ will not be independent of γ , m . However, if l_{cutoff} is sufficiently large, then

$$|\nabla \cdot \kappa_e^{\gamma,l,m} \nabla T|_{\gamma,l,m} \ll |[(\hat{v} \cdot \nabla)\hat{T}]_{\gamma,l,m}| \quad \text{for } l \ll l_{\text{cutoff}}. \quad (2.14)$$

Therefore, small errors in eddy diffusivity caused by assuming that $\kappa_e^{\gamma,l,m}$ is independent of γ and m for low values of l will be negligible in comparison to the explicitly calculated nonlinear advective terms.

Assuming that the eddy diffusivity and viscosities are independent of γ and m , we parametrize them as

$$\kappa_e = 2\pi [l_{\text{cutoff}}(l_{\text{cutoff}} + 1)]^{-1/2} [\text{KE}(l_{\text{cutoff}}, r)/2\pi r^2]^{1/2} C_k, \quad (2.15)$$

$$v_p = 2\pi [l_{\text{cutoff}}(l_{\text{cutoff}} + 1)]^{-1/2} [\text{KE}(l_{\text{cutoff}}, r)/2\pi r^2]^{1/2} C_p, \quad (2.16)$$

$$v_T = 2\pi [l_{\text{cutoff}}(l_{\text{cutoff}} + 1)]^{-1/2} [\text{KE}(l_{\text{cutoff}}, r)/2\pi r^2]^{1/2} C_T, \quad (2.17)$$

where $\text{KE}(l, r)$ is the kinetic energy of all modes with wavenumber l at radius r (see Marcus 1980a, hereafter Paper II). The length and velocity at the resolution limit are taken to be $2\pi [l_{\text{cutoff}}(l_{\text{cutoff}} + 1)]^{-1/2}$ and $[\text{KE}(l_{\text{cutoff}}, r)/2\pi r^2]^{1/2}$, respectively. The eddy coefficients C_p , C_T , and C_k are functions of l as well as r and t . However, if l_{cutoff} lies in the self-similar inertial range, dimensional arguments can be used to show that C_k , C_T , and C_p should be independent of l . The gist of the argument is that the rate at which energy is transferred from modes of wavenumber l to modes with wavenumber greater than l_{cutoff} is (Tennekes and Lumley 1972)

$$S(l)^2 E(l_{\text{cutoff}}) / S(l_{\text{cutoff}}), \quad (2.18)$$

where $S(l)$ is the strain and $E(l)$ is the energy associated with modes of wavenumber l . If $v \sim l^{-1/3}$, then $S \sim l^{2/3}$ and $E \sim l^{-2/3}$ so the rate is proportional to $(l/l_{\text{cutoff}})^{4/3}$. Using the eddy viscosity in the modal Navier-Stokes equation, this same dissipation rate is proportional to

$$l^2 E(l) v_p^{\gamma,l,m} = C_p (l/l_{\text{cutoff}})^{4/3} \quad (2.19)$$

if $C_p \approx C_T$. Setting the two expressions for the rates equal, we see that C_p is independent of l .

c) Self-Consistency and a Method of Solution

The most difficult task is implementing the eddy viscosity is choosing the values of C_p , C_T , and C_k . They are usually chosen (Deardorff 1972; Siggia and Patterson 1978) by varying their values until the resulting kinetic and thermal energy spectra have smooth, $k^{-5/3}$, power laws. This method is not only arbitrary but also somewhat expensive in that several solutions with differing values of the eddy coefficients must be computed before the correct values of the eddy coefficients can be determined.

We shall establish the values of the eddy coefficients by appealing to the self-consistency requirement that C_p , C_T , and C_k are independent of l_{cutoff} . Specifically, the time-averaged thermal energy, poloidal kinetic energy, and toroidal kinetic energy spectra should not change when l_{cutoff} is changed from 12 to 10. A necessary condition for the thermal energy spectra to be the same is that the rate at which the thermal energy cascades in and out of the modes with wavenumber l due to the advective term is independent of l_{cutoff} . The rate of this energy cascade is

$$\sum_{\gamma,m} \hat{T}_{\gamma,l,m} [(\mathbf{v} \cdot \nabla)T]_{\gamma,l,m}.$$

For the cascade rate with $l_{\text{cutoff}} = 12$ to be equal to the rate with $l_{\text{cutoff}} = 10$ requires that

$$\hat{T}_{\gamma,l,m} \nabla \cdot \{ [\kappa_e^{\gamma,l,m}(12) - \kappa_e^{\gamma,l,m}(10)] \nabla \hat{T} \}_{\gamma,l,m} = \hat{T}_{\gamma,l,m} \{ \mathbf{v} \cdot \nabla [T_{11} + T_{12}] + [\mathbf{v}_{11} + \mathbf{v}_{12}] \cdot \nabla \hat{T} + [\mathbf{v}_{11} + \mathbf{v}_{12}] \cdot \nabla [T_{11} + T_{12}] \}_{\gamma,l,m}, \quad (2.20)$$

where $\kappa_e^{\gamma,l,m}(12)$ and $\kappa_e^{\gamma,l,m}(10)$ are calculated from equation (2.15) using l_{cutoff} equal to 12 and 10, respectively, and where \hat{T} and $\hat{\mathbf{v}}$ are the resolvable temperature and velocity with $l_{\text{cutoff}} = 10$. Using equation (2.15) for $\kappa_e^{\gamma,l,m}$ in equation (2.20), we obtain a linear differential equation for $C_k(r)$ which may be numerically solved:

$$C_k \left[A(r) \sum_{\gamma,m} T_{\gamma,l,m} \mathcal{D}_l T_{\gamma,l,m} + \frac{1}{2} (\partial A / \partial r) \partial T^2_{\gamma,l,m} / \partial r \right] + \frac{1}{2} \frac{\partial C_k}{\partial r} A(r) \sum_{\gamma,m} \partial T^2_{\gamma,l,m} / \partial r = \sum_{\gamma,m} T_{\gamma,l,m} [\hat{\mathbf{v}} \cdot \nabla (T_{11} + T_{12}) + (\mathbf{v}_{11} + \mathbf{v}_{12}) \cdot \nabla (\hat{T} + T_{11} + T_{12})]_{\gamma,l,m} \quad (2.21)$$

where

$$A(r) \equiv (2\pi)^{1/2} r^{-1} [(12 \cdot 13)^{-1/2} \text{KE}(12, r)^{1/2} - (10 \cdot 11) \text{KE}(10, r)^{1/2}] \quad (2.22)$$

and where \mathcal{D}_l is the two-dimensional Laplacian, $\mathcal{D}_l \equiv (\nabla^2 - 1/r \partial^2 / \partial r^2)$. Now, C_k obtained from equation (2.21) will obviously be a function of r . Also, we can obtain different values of C_k by choosing a different value of l in equation (2.21) or by choosing a different pair of values of l_{cutoff} . It is a test of the self-consistency of our eddy diffusivity model if we can show that C_k does not vary much with radius or depend strongly on our choices of l and l_{cutoff} . By examining the kinetic energy budgets of the poloidal and toroidal velocity, we obtain differential equations similar to (2.21) for C_p and C_T .

An advantage of using equation (2.21) to compute C_k at each time step is that the iterative values converge rapidly. To see the rapid convergence, let us assume that we have underestimated C_k . A small value of C_k makes the thermal energy extraordinarily large at the $l = 11$ and $l = 12$ wavenumbers since it is not dissipated and piles up near l_{cutoff} (see Marcus 1980*b*). The upward curl of the energy spectrum makes the right-hand side of equation (2.21) large and thereby increases C_k . The convergence of the equations that govern C_p and C_T is even more rapid because it can be shown that C_p and C_T are nearly proportional to $1/A(r)$ or $1/\{[\text{KE}(10, r)]^{1/2} - 0.8397[\text{KE}(12, r)]^{1/2}\}$. If C_p or C_T is underestimated, the upward curl on the energy spectrum at $l = 12$ causes this denominator to shrink, causing C_p and C_T to increase rapidly. Likewise, if C_p is too large, $\text{KE}(12, r)$ dissipates to zero, the denominator becomes large, and C_p decreases.

d) Numerical Results

We have integrated the modal equations of convection (Paper II) with the eddy viscosities and conductivity described in this section. For a Prandtl number of 1 and a Rayleigh number less than 10^8 , the values obtained for the eddy coefficients $C_p(r)$, $C_T(r)$, and $C_k(r)$ varied widely with radius, l , and pair of l_{cutoff} 's. We have concluded that our model of the eddy viscosity is not valid in this regime. This result should not be surprising since the Reynolds numbers are less than 500 and the high wavenumber modes are neither isotropic nor lie in an inertial range.

For a Rayleigh number of 10^{10} the eddy coefficients do converge. With l_{cutoff} equal to 10 and 12 and l equal to 8 in equation (2.20), the value of C_k changes by less than 15% from $r = 0$ to $r = 0.8$. The time-averaged value of C_p averaged over $0 \leq r \leq 0.8$ is 0.52. In the turbulent boundary layer between $r = 0.8$ and $r = 1.0$, C_k is $\sim 20\%$ lower than this value. The coefficients C_T and C_p are similarly well behaved with average values of 0.55 and 0.46, respectively. Using the velocity and temperature produced from these eddy coefficients in equation (2.21) with $l = 6$ and l_{cutoff} equal to 10 and 8, we have computed a new averaged value of C_k that is $\sim 10\%$ lower than the old value. The decrease in eddy diffusivity is due to the fact that the calculated $l = 6$ energies are higher (due to production) than we would have predicted by using the value of the thermal energy at $l = 12$ and extrapolating to $l = 6$ using a $k^{-5/3}$ spectrum (see § III). New average values of C_p and C_T were also computed based on $l = 6$ and $l_{\text{cutoff}} = 10$ and 8. The value of C_p was $\sim 10\%$ lower, but C_T was $\sim 10\%$ higher than its original value because the $l = 6$ toroidal kinetic energy is lower than we would have predicted (assuming a $k^{-5/3}$ spectrum and knowing the value of the toroidal kinetic energy at $l = 12$). The toroidal kinetic energy is relatively low at low wavenumbers because it is not produced at the low wavenumbers by buoyancy; it is produced at the high wavenumbers by isotropization of the velocity. The new values of C_p , C_T , and C_k , like the old values, are only weakly ($\sim 12\%$) dependent upon radius. We have computed the time-averaged eddy coefficients with three other possible combinations of l and l_{cutoff} and find only a weak ($\sim 15\%$) dependence on these values. Because the eddy coefficients are nearly independent of l , r , and the pair of l_{cutoff} 's, we conclude that our choices of the functional forms of the eddy viscosities and diffusivity are self-consistent for $\text{Rs} = 10^{10}$. It is also reassuring that the three eddy coefficients are nearly the same and are all of order unity. For the remainder of the numerical results presented in this paper, we have held C_p , C_T , and C_k fixed at their averaged values of 0.46, 0.55, and 0.52, respectively.

By holding the eddy coefficients fixed, the eddy viscosities and diffusivity are functions of the time-averaged, local, rms velocity field. From equation (2.15) we compute the total diffusivity, $\kappa_T = \kappa_e + \kappa$ (where κ is the molecular diffusivity), by setting $l_{\text{cutoff}} = 12$, $C_k = 0.52$, and evaluating the time-averaged kinetic energy for $l = 12$ (see Fig. 5):

$$\kappa_T = \kappa[1 + 31.5f(r)]. \quad (2.23)$$

The dimensionless function $f(r)$ is $\text{KE}(l = 12, r)r^{-2}$ normalized so that its maximum value is unity; $f(r)$ monotonically increases with radius, and $f(0) = 0$. With a similar expression for the viscosity we calculate the effective Rayleigh number, Re , which is computed with ν_T and κ_T instead of ν and κ . For an actual Rayleigh number of 10^{10} the effective Rayleigh number is

$$\text{Re} \approx 3.20 \times 10^5 f(r)^{-3}. \quad (2.24)$$

This means that Re decreases from 10^{10} at $r = 0$ (discounting the effects of numerical viscosity) to 3.2×10^5 at the boundary.

III. TIME-AVERAGED NUMERICAL RESULTS

a) Temperature and Flux

We have computed a numerical solution for convection in a sphere with a Rayleigh number, Rs , of 10^{10} , and a Prandtl number, Pr , of 1, and with the boundary conditions and dimensionless units described in Paper II. The heat source, $H(r)$, is constant for $r < 0.3$ and zero outside this radius. We have used the time-averaged eddy coefficients described in § II. Although the convective solution is time-dependent, the energy spectra are time-independent when averaged over several of the largest eddy turnover times; the results presented in this section have been averaged over ~ 30 eddy turnover times. We use the notation that single brackets, $\langle \rangle$, around a quantity indicate an average over θ , ϕ ; and double brackets, $\langle\langle \rangle\rangle$, indicate an average over θ , ϕ , and t . The horizontal and time-averaged temperature as a function of radius, $\langle\langle T(r) \rangle\rangle$, is shown in Figure 1. The fluid is nearly isothermal except at the outer boundary which has a thickness of ~ 0.03 . By an asymptotic expansion it can be shown that $|\partial T/\partial r| \sim (\text{Rs})^{-1/3} \approx 5 \times 10^{-4}$ far away from the boundary layer. The time-averaged numerical results are not inconsistent with this value, but at any particular value of radius and time the fluctuation in $|\partial T/\partial r|$ can be 10–100 times this value (see § III). The central temperature is $\langle\langle T(0) \rangle\rangle = 0.036$, which should be compared to the central temperature of 0.68 calculated with $\text{Rs} = 10^4$, $\text{Pr} = 10$ (Paper II), and a central temperature of 4.0 for a fluid in conductive equilibrium. Although $|\partial T/\partial r|$ scales as $(\text{Rs})^{-1/3}$, the central temperature scales as $|\partial T/\partial r| \times \text{boundary-layer thickness}$. Since the turbulent boundary-layer thickness has a slight Rayleigh number dependence, $\langle\langle T(0) \rangle\rangle$ scales only approximately as $(\text{Rs})^{-1/3}$. Figure 1 shows a slight bump at $r = 0.3$ which is the boundary of the heat source. This bump is not nearly as prominent as it was for $\text{Rs} = 10^4$, and we conclude that at large Rayleigh numbers the convective velocity is very efficient in smoothing out the temperature perturbations due to the inhomogeneities in the heat source. The mean temperature curve also shows that the temperature gradient, $\langle\langle \partial T/\partial r \rangle\rangle$, is positive near the origin and near the outer boundary layer. In these two regions, the conductive flux transports energy toward the center of the star; the time-averaged convective flux due to all modes, $\text{Total } F_{\text{con}}$, is greater than the steady-state flux of the heat source,

$$F_{\text{ss}}(r) = (4\pi r^2)^{-1} \int_0^r H(r') dr'.$$

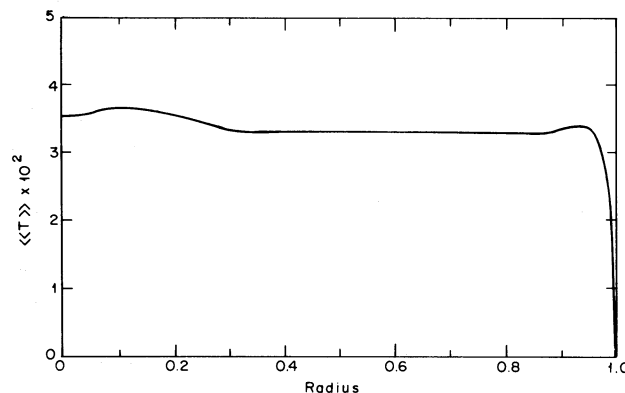


FIG. 1.—The time-averaged mean temperature as a function of radius. The break in the curve at $r = 0.3$ marks the boundary of the internal heat source.

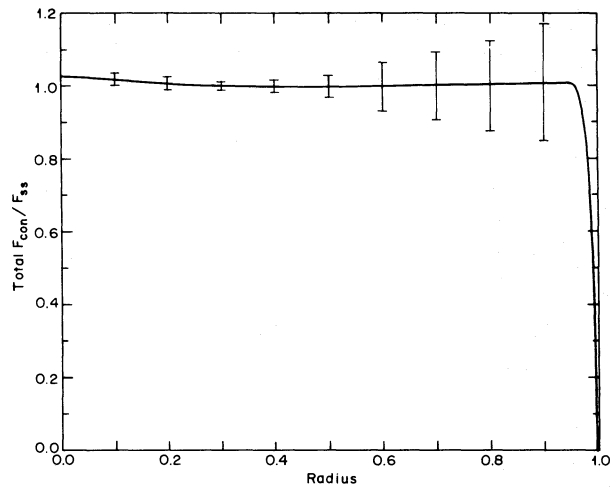


FIG. 2.—The ratio of the time-averaged convective flux carried by all of the modes to the steady-state flux due to the internal heat source as a function of radius. Overshooting occurs near the origin and boundary layer. The variances of the ratio of fluxes are plotted with error bars and reach a maximum near the boundary layer.

We have plotted the ratio of these two fluxes in Figure 2. Far away from the boundary layer this ratio should be $\sim 1 - (Rs)^{-1/3}$. Convective overshoot is indicated where $\text{Total } F_{\text{con}}/F_{\text{ss}}$ is greater than 1. Because $\text{Total } F_{\text{con}} \approx F_{\text{ss}}$ and is so much greater than the conductive flux, there can be large (10–100) fluctuations in the conductive flux (i.e., temperature gradient) without affecting the total flux. Plotted on the curve in Figure 2 with error bars is the variance of $\text{Total } F_{\text{con}}/F_{\text{ss}}$ (the variance σ of a quality x is with respect to time, $\sigma \equiv [\langle\langle(x - \langle\langle x \rangle\rangle)^2\rangle\rangle]^{1/2}$). The variance of the total (convective plus conductive) flux divided by F_{ss} is nearly identical to the variance shown in Figure 2. The increase in σ with radius is due to the fact that the local Rayleigh number in the surface boundary layer is near its critical value. Secondary convective instabilities in the boundary layer cause time-dependent bursts in the convective flux. These fluctuations propagate inward and diminish in amplitude as they approach the origin. Notice that the variance at the surface is $\sim 18\%$ of the total flux.

b) Two- and Three-Dimensional Spectra

The time-averaged kinetic and fluctuating thermal energy spectra (see Paper II for definitions) at $r = 0.5$ as functions of the two-dimensional wavenumber, l , are shown in Figure 3. Both spectra behave as power laws at large l , but for $l \leq 6$ the kinetic energy spectrum is nearly exponential and obviously shows the effects of production. The thermal energy spectrum has a noticeable “cliff” between $l = 4$ and $l = 5$. Each spectrum is normalized by the total energy at $r = 0.5$ (due to all modes).

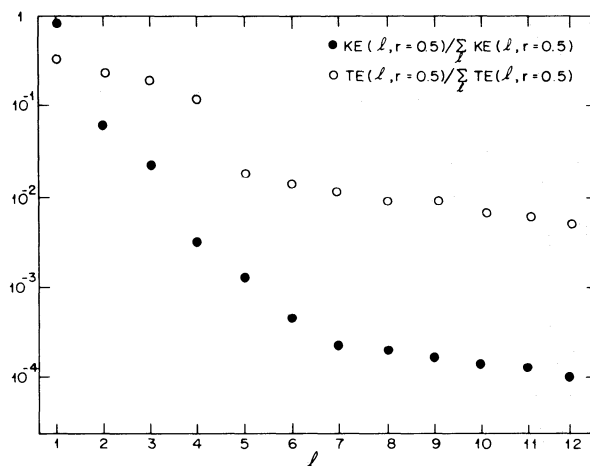


FIG. 3.—The time-averaged two-dimensional spectra of kinetic and fluctuating thermal energy as a function of l at $r = 0.5$. For $l \geq 7$, the spectra behave as power laws.

In dimensionless units the kinetic energy (which scales as $Rs^{2/3}$) at $r = 0.5$ is

$$\sum_l KE(l, r = 0.5) = 6.26 \times 10^7.$$

Approximately 91% of this energy is in the $l = 1$ modes. The fluctuation thermal energy (which scales as $Rs^{-2/3}$) at $r = 0.5$ is 4.5×10^{-6} , and approximately 35% of this energy is in the $l = 1$ modes. The variances of the $l = 1$ component of the kinetic and thermal energies are 0.08 $KE(1, 0.5)$ and 0.2 $TE(1, 0.5)$, respectively. The energies of the three $l = 1$ modes ($Y^{R,1,0}$, $Y^{R,1,1}$, and $Y^{I,1,1}$) completely change in one eddy turnover time. If the three modes were statistically independent, then the $l = 1$ kinetic and thermal energy would each have a fractional variance of $1/\sqrt{3}$. Because our numerically determined variances are much smaller, the three $l = 1$ components of the kinetic and thermal energies must be very well correlated with each other. The two-dimensional kinetic energy spectrum has been deconvolved into a three-dimensional spectrum and plotted as a function of the log of the three-dimensional wavenumber, k , in Figure 4. The dashed line is $k^{-5/3}$ and is provided for comparison with a Kolmogorov spectrum. The spectrum clearly shows a production peak at the low wavenumbers, a cliff at $k \approx 3$, and an equilibrium range for $k > 6$. The wavenumber of the Kolmogorov length calculated from rate of kinetic energy dissipation (§ III d) is $\sim 3 \times 10^3$. As discussed in Paper II, approximations are introduced into the transformation of a two-dimensional spectrum into a three-dimensional spectrum, so that 5/3-power-law behavior of Figure 4 should be viewed with some caution.

The kinetic energy $KE(l, r)$ as a function of radius is graphed for each value of l in Figure 5. The curves are qualitatively the same as they were for $Rs = 10^4$, $Pr = 10$ (Paper II). Except for $l = 1$ and $l = 12$, each curve is zero at the origin, rises to a peak at $0.5 < r < 0.8$, decreases to a local minimum, and then increase sharply in the outer boundary layer. The peak shifts to larger values of radius with increasing l . Using the distance between the local minimum of $KE(l, r)$ and $r = 1.0$ as the turbulent boundary-layer thickness, we see that the thickness is fairly constant for $l > 6$. The turbulent boundary layer is due to our eddy viscosity. The viscous sublayer, with a thickness of $\sim KE(l, r = 1)^{-1/2}$, is unresolvable. For $l = 12$, the local minimum in $KE(l, r)$ denotes that the turbulent boundary layer has disappeared. The disappearance is because our eddy viscosity dominates the $l = 12$ mode not just in the boundary layer but for all values of r .

The time-averaged fraction of the total convective flux at $r = 0.5$ that is carried by modes of wavenumber l , $F_{con}(l)/Total F_{con}$, is plotted in Figure 6. Far from being a smooth power-law spectrum, we see that: (1) over 90% of the convective flux is carried by the $l = 1$ modes. (2) virtually no flux is carried by modes with $l \geq 6$, and (3) the $l = 3$ modes advect energy downward. The convective flux carried by the modes of wavenumber l is

$$\sum_{\gamma, m} \langle (e_r \cdot v)_{\gamma, l, m} T_{\gamma, l, m} \rangle.$$

We therefore expect $F_{con}(l)/Total F_{con}$ to be proportional to the square root of the product of the kinetic and thermal energy spectra. We also expect $F_{con}(l)$ to be proportional to the correlation between the radial velocity and temperature:

$$\delta(l, r) \equiv \sum_{\gamma, m} (e_r \cdot v)_{\gamma, l, m} T_{\gamma, l, m} / \left\{ \sum_{\gamma, m} [\hat{e}_r \cdot v]_{\gamma, l, m}^2 \sum_{\gamma, m} [T_{\gamma, l, m}]^2 \right\}^{1/2}. \tag{3.1}$$

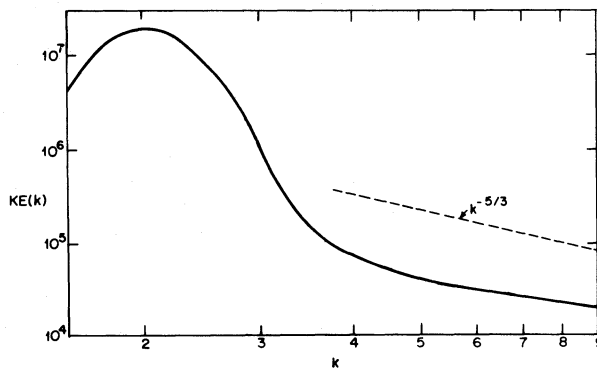


FIG. 4.—The time-averaged kinetic energy spectrum as a function of the three-dimensional wavenumber, k . The dashed line has a slope of $-5/3$ and corresponds to a Kolmogorov spectrum.

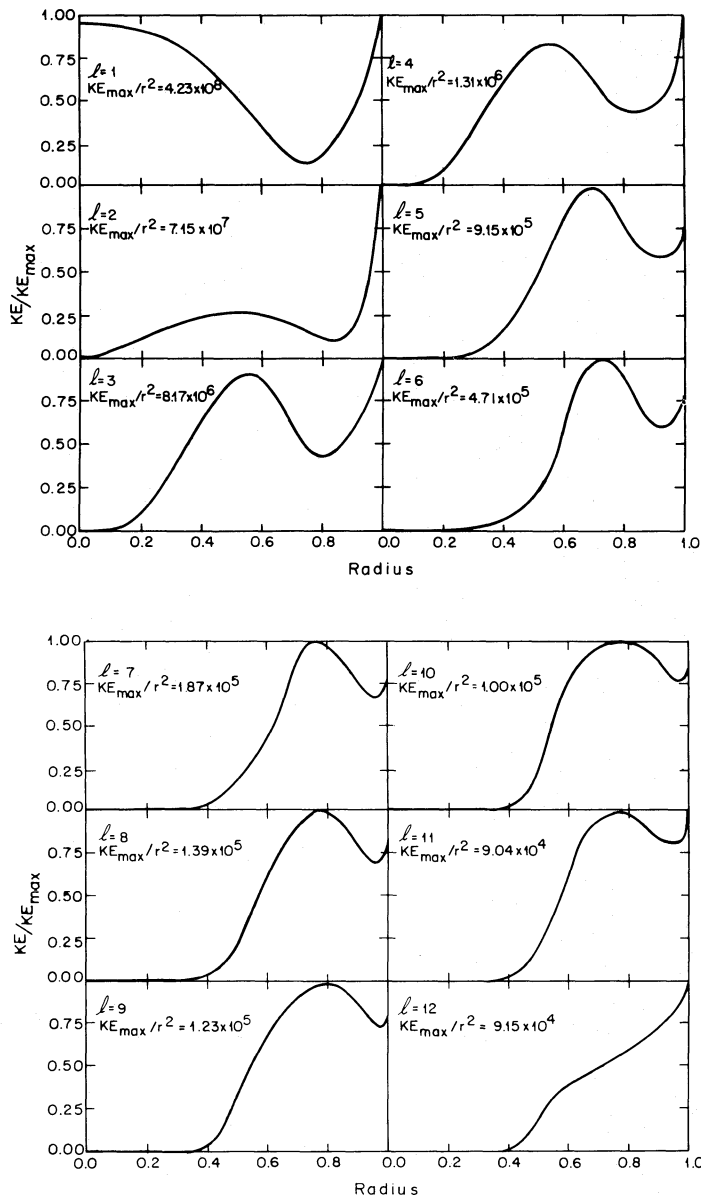


FIG. 5.—The time-averaged kinetic energy spectra $KE(l, r)/r^2$ as functions of radius for $l = 1, 12$. The spectra are similar to the results obtained with $Rs = 10^4$, $Pr = 10$ in that the radius at which the peak of each spectrum occurs increases with l . Each spectrum is normalized by its maximum value, KE_{\max}/r^2 .

We have found that there is a slow, systematic decrease in $\delta(l, r)$ with increasing l . The decrease means that the large-scale velocity and temperature have better *spatial* correlation than the small-scale velocity and temperature. The time-averaged correlation, $\langle\langle\delta(l, r)\rangle\rangle$, decreases more rapidly with l than does $\langle\delta(l, r)\rangle$, indicating that the large-scale velocity and temperature modes are also better correlated in *time* than are the small-scale modes. Note that $\langle\langle\delta(l = 3, r = 0.5)\rangle\rangle$ is less than zero because the $l = 3$ convective flux is negative. The time-averaged total correlation of the radial velocity and temperature for all l is

$$\langle\langle\delta(r)\rangle\rangle \equiv \langle\langle[\langle v_r \tilde{T} \rangle / (\langle v_r^2 \rangle \langle \tilde{T}^2 \rangle)^{1/2}] \rangle\rangle. \quad (3.2)$$

In Figure 7 we have plotted $\langle\langle\delta(r)\rangle\rangle$ as a function of radius. The correlation varies between 0.5 and 1.0. In comparison, Deardorff and Willis (1967) find that in plane-parallel convection in air ($Pr \approx 0.7$) with a Rayleigh

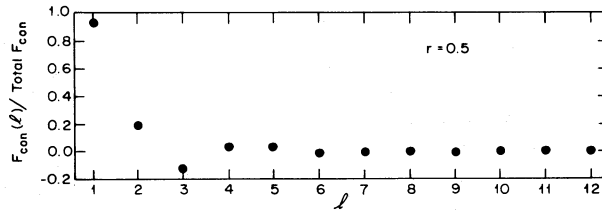


FIG. 6.—The ratio of the convective flux carried by modes of wavenumber l to the total convective flux carried by all modes. The figure is computed at $r = 0.5$. The negative value of the ratio at $l = 3$ indicates that the $l = 3$ modes are performing mechanical work on the fluid.

number of 10^7 , the correlation $\delta(r)$ varies between 0.5 and 0.7. They also find a decrease in the correlation, δ , with wavelength.

c) Isotropy

As in Paper II, we have used $I(l, r) \equiv \text{KE}_H(l, r)/2\text{KE}_R(l, r)$ as a measure of the isotropy at the flow, where $\text{KE}_R(l, r)$ is the kinetic energy in the radial direction at r due to motions of wavenumber l , and where $\text{KE}_H(l, r)$ is the kinetic energy in the horizontal direction. When $I(l, r) \approx 1$, the flow is isotropic. In Paper II, it was shown that for all l , $I(l, r)$ is constrained to be equal to 1 at $r = 0$. Furthermore, in the outer boundary layer, the radial component of the velocity goes to zero, so $I(l, r)$ goes to infinity. The ratio, $\text{KE}_H(l, r)/2\text{KE}_R(l, r)$, is a useful measure of isotropy only for $0.2 \lesssim r \lesssim 0.8$. The isotropy is plotted as a function of r for all l in Figure 8. The $l = 1$ isotropy is qualitatively similar to the values calculated with $\text{Rs} = 10^4$, $\text{Pr} = 10$ (Paper II); the radial component of the velocity dominates except in the boundary layer. For $\text{Rs} = 10^4$ we found that the radial velocity was dominant for all l . Figure 8 shows that for $l \geq 3$ the horizontal component of the velocity becomes important at some values of the radius. More importantly, for large values of l , $[I(l, r) - 1]$ becomes small for $r < 0.8$, indicating that the flow becomes isotropic at large wavenumbers. Note that $I(l, r)$ for $l = 12$ increases monotonically from $r = 0.2$ to $r = 0.8$; this behavior is due to the eddy viscosity.

d) Energy Budgets

We find that the time-averaged rate at which fluctuating thermal energy enters the fluid due to all modes, TE_{in} , is 0.166, and the time-averaged rate at which kinetic energy enters the fluid due to all modes, KE_{in} , is 6.01×10^{10} . In Paper II we showed that KE_{in} for an isothermal fluid is $1.892\pi \times \text{PrRs} \approx 5.9 \times 10^{10}$; we see that the calculated and isothermal values of KE_{in} are nearly the same. In Figure 9 we have plotted $\text{KE}_{in}(l)$ and $\text{TE}_{in}(l)$, the rates at which kinetic and fluctuating thermal energy obtained directly from buoyancy (not from cascading) enter the shell of modes with wavenumber l . The definitions $\text{TE}_{in}(l)$ and $\text{KE}_{in}(l)$ in terms of modes appear in Paper II. Figure 9 shows that $\sim 87\%$ of KE_{in} and $\sim 74\%$ of TE_{in} are due to the $l = 1$ modes. Note that $\text{KE}_{in}(l = 3)$ is negative, which means that the $l = 3$ velocity and temperature fluctuations are *negatively* correlated [i.e., $\delta(l = 3, r) < 0$]. For $l > 3$, $\text{KE}_{in}(l)/\text{KE}_{in}$ and $\text{TE}_{in}(l)/\text{TE}_{in}$ are both much less than 1, so the production part of the spectrum is confined to $l \lesssim 3$.

The rate at which fluctuating energy cascades into the set of modes with wavenumber l is

$$\text{TE}_{cas}(l) = -4\pi \sum_{\gamma, m} \int_0^1 \langle (\mathbf{v} \cdot \nabla T)_{\gamma, l, m} T_{\gamma, l, m} \rangle r^2 dr.$$

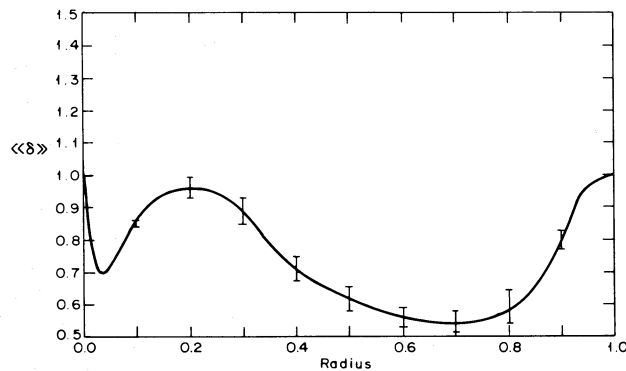


FIG. 7.—The time-averaged correlation of the radial velocity and temperature $\langle\langle \delta \rangle\rangle \equiv \langle\langle (v_r T) / (v_r^2)^{1/2} (T^2)^{1/2} \rangle\rangle$. The variance of the correlation is denoted by error bars.

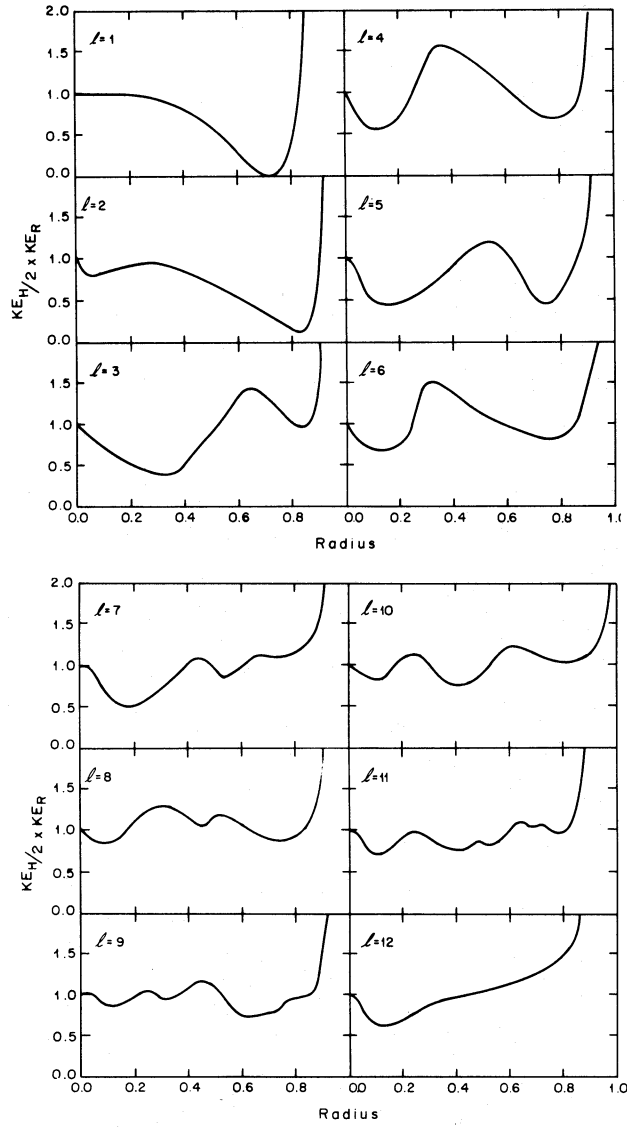


FIG. 8.—The ratio of the horizontal kinetic energy to twice the radial kinetic energy as a function of radius for each wavenumber l . By definition, the ratio is 1 at the origin and goes to infinity in the boundary layer where the radial component of the velocity vanishes. The ratio is a good measure of the isotropy for $0.2 \lesssim r \lesssim 0.8$. As l increases, the ratio goes to unity, demonstrating that the flow becomes nearly isotropic at small wavelengths.

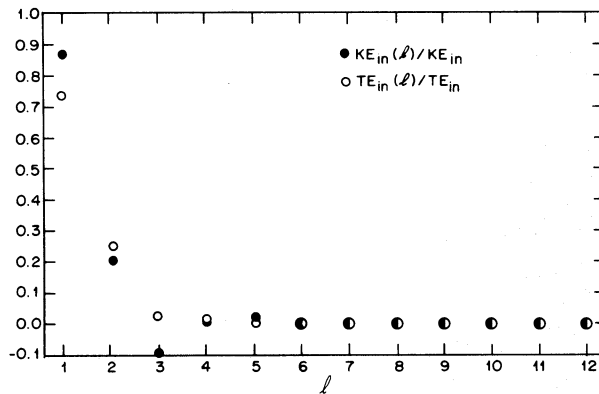


FIG. 9.—The fraction of the total amount of kinetic (thermal) energy generated that is produced by modes of wavenumber l is shown by solid (open) circles. Negligible energy production occurs when $l \gtrsim 6$.

We can arbitrarily write this rate as the difference

$$TE_{\text{cas}}(l) = TE_{\text{up}}(l) - TE_{\text{down}}(l), \quad (3.3)$$

where

$$TE_{\text{up}}(l) = -4\pi \sum_{\gamma,m} \int_0^1 \langle (\mathbf{v} \cdot \nabla T_{<})_{\gamma,l,m} T_{\gamma,l,m} \rangle r^2 dr, \quad (3.4)$$

$$TE_{\text{down}}(l) = 4\pi \sum_{\gamma,m} \int_0^1 \langle (\mathbf{v} \cdot \nabla T_{>})_{\gamma,l,m} T_{\gamma,l,m} \rangle r^2 dr, \quad (3.5)$$

and where

$$T_{>} = \sum_{l' > l} T_{l'} \quad (3.6)$$

and

$$T_{<} = \sum_{l' < l} T_{l'}. \quad (3.7)$$

Physically, we can think of $TE_{\text{up}}(l)$ as the rate at which the thermal energy cascades into the modes with wavenumber l due to thermal fluctuations larger than the wavelength of the l -modes, and $TE_{\text{down}}(l)$ as the rate at which the thermal energy of the modes of wavenumber l feeds the smaller thermal fluctuations. If there were no dissipation, then

$$TE_{\text{in}}(l) + TE_{\text{up}}(l) = TE_{\text{down}}(l). \quad (3.8)$$

We plot $TE_{\text{up}}(l)/TE_{\text{down}}(l)$ in Figure 10 and use this ratio to determine whether or not the wavenumber l belongs in the production or inertial range. For $l = 0$, $TE_{\text{up}}(l)$ must equal zero. As l increases, the ratio goes smoothly to 1. For $l \geq 6$, $TE_{\text{up}}(l)/TE_{\text{down}}(l) \approx 1$, which indicates that $TE_{\text{in}}(l) \ll TE_{\text{up}}(l)$ or equivalently that the modes with wavenumber l derive most of their energy from the cascade and not from direct production; l lies in the inertial range for $l \geq 6$. We can define $KE_{\text{up}}(l)$ and $KE_{\text{down}}(l)$ in a fashion similar to equations (3.4) and (3.5). The ratio $KE_{\text{up}}(l)/KE_{\text{down}}(l)$ shown in Figure 10 is near unity for $l > 6$, indicating that the kinetic energy spectrum is also inertial for $l \geq 6$. For $l = 3$ the ratio of $KE_{\text{up}}(l)$ to $KE_{\text{down}}(l)$ is anomalously large because $KE_{\text{in}}(l = 3)$ is negative (see Fig. 9).

Because we are numerically limited to a finite resolution, we can only approximate nonlinear quantities such as $TE_{\text{up}}(l)$ and $TE_{\text{down}}(l)$. The approximation used in evaluating $TE_{\text{up}}(l)$ is

$$\langle T_{\gamma,l,m} (\mathbf{v} \cdot \nabla T_{<})_{\gamma,l,m} \rangle \approx \langle T_{\gamma,l,m} (\hat{\mathbf{v}} \cdot \nabla T_{<})_{\gamma,l,m} \rangle, \quad (3.9)$$

where $\hat{\mathbf{v}}$ is the numerically resolvable part of \mathbf{v} (see § II). Using the eddy diffusivity, $\kappa_e^{\gamma,l,m}$, that was introduced in § II, we compute $TE_{\text{down}}(l)$ with the approximation

$$\int_0^1 r^2 dr \langle T_{\gamma,l,m} (\mathbf{v} \cdot \nabla T_{>})_{\gamma,l,m} \rangle \approx \int_0^1 r^2 dr \{ \langle T_{\gamma,l,m} (\hat{\mathbf{v}} \cdot \nabla \hat{T}_{>})_{\gamma,l,m} \rangle - \langle T_{\gamma,l,m} \nabla \cdot [\kappa_e^{\gamma,l,m} \nabla (T_{\gamma,l,m} Y^{\gamma,l,m})] \rangle \}, \quad (3.10)$$

where $\hat{T}_{>}$ is the resolvable part of $T_{>}$. The approximation in (3.11) is based on both the definition of $\kappa_e^{\gamma,l,m}$ and the

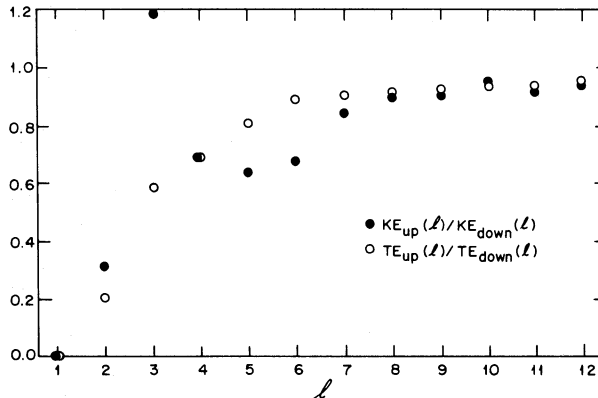


FIG. 10.—The ratio of the rate at which kinetic (thermal) energy cascades into modes of wavenumber l from all larger wavelength modes to the rate at which kinetic (thermal) energy cascades from the modes of wavenumber l to all smaller wavelength modes. By definition the ratio must be zero for $l = 1$. For $l > 6$ the ratio approaches unity and is indicative of an inertial range where production and dissipation are unimportant.

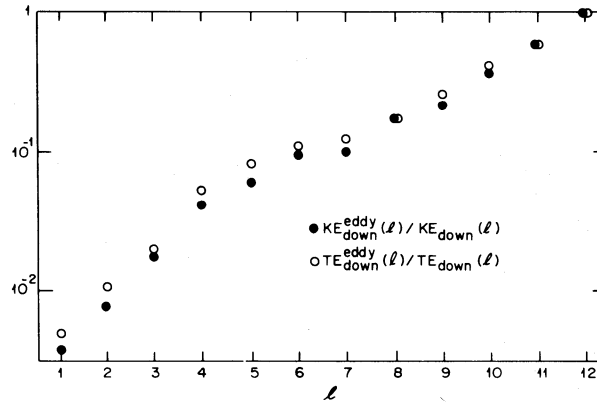


FIG. 11.—The ratio of the rate at which kinetic (thermal) energy cascades from modes of wavenumber l to wavelengths smaller than the limit of numerical resolution to the rate at which the energy cascades from l to all smaller wavelength modes. The rate at which the energy cascades to the modes smaller than the resolution limit is computed with the eddy viscosity (diffusivity). The plotted ratio shows that only modes with $l > 9$ directly feel the eddy viscosity (diffusivity).

assumption that the energy cascade is local in wavenumber space. We see from (3.11) that two terms contribute to $TE_{\text{down}}(l)$: one due to the resolvable part of the temperature and the other to the unresolvable part that is parametrized by the eddy diffusivity.

In Figure 11 we plot the fraction of $TE_{\text{down}}(l)$ that is due to the eddy diffusivity, $TE_{\text{down}}^{\text{eddy}}(l)/TE_{\text{down}}(l)$, as a function of l . Figure 11 shows that the eddy diffusivity has a negligible contribution to $TE_{\text{down}}(l)$ for $l < 9$. This means that the eddy diffusivity has little effect on the large-scale modes; it directly affects only the small-scale modes. The large-scale modes feel the eddy diffusivity only through the nonlinear interactions with the small scale modes. In Figure 11 we have also plotted the ratio $KE_{\text{down}}^{\text{eddy}}(l)/KE_{\text{down}}(l)$ that is the fraction of $KE_{\text{down}}(l)$ due to the unresolvable part of v parametrized through the eddy viscosities. Again, it is only the small-scale modes that directly feel the eddy viscosities.

IV. TIME-DEPENDENT RESULTS

a) Instantaneous Velocity and Temperature

The velocity and temperature are time-dependent and are self-correlated for approximately one eddy turnover time. We estimate the turnover time of the largest eddies as $2 \times \langle\langle v^2(r=0) \rangle\rangle^{-1/2} \approx 3 \times 10^{-4}$. When averaged over many turnover times, v and \tilde{T} are zero. The $Y^{R,1,0}$ components of the temperature, poloidal, and toroidal velocities at one particular instant in time are plotted in Figures 12, 13, and 14, respectively. The plotted functional forms of the $l = 1$ temperature and poloidal velocity are characteristic of the forms for all values in time; it is only the amplitude of the functional form of the largest $l = 1$ mode that changes rapidly in time as the three $l = 1$ modes

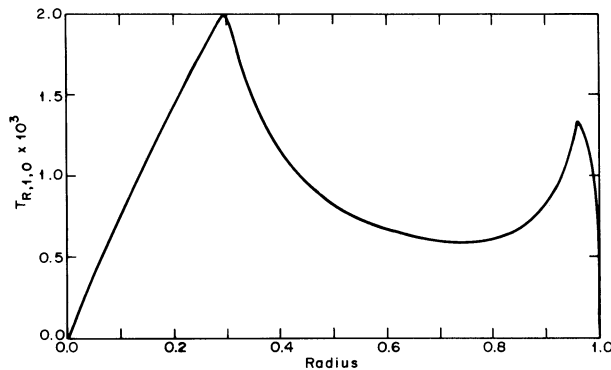


FIG. 12

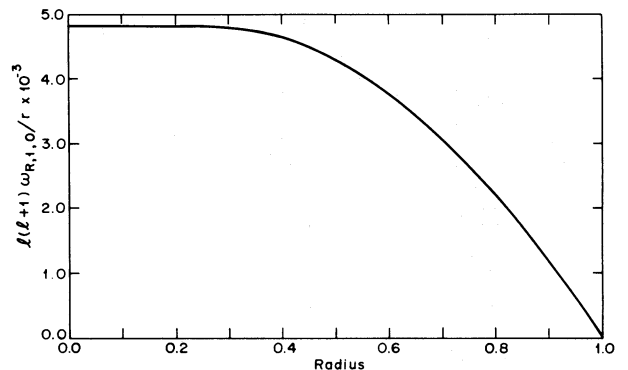


FIG. 13

FIG. 12.—The instantaneous temperature of the $Y^{R,1,0}$ mode as a function of radius. The functional form of this mode with one peak at $r = 0.3$ and the other at the boundary layer is typical of the largest $l = 1$ mode.

FIG. 13.—Same as Fig. 12, except the radial component of the velocity, $l(l+1)\omega_{R,1,0}/r$, is shown here.

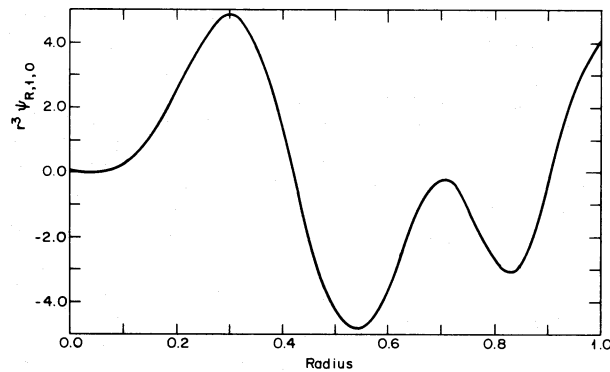


FIG. 14.—Same as Fig. 13, except with the toroidal component of the velocity. The angular momentum about the z -axis is $4\pi r^3 \psi_{R,1,0}$, so $\int_0^1 r^3 \psi_{R,1,0} dr = 0$. The functional form of $r^3 \psi_{R,1,0}$ changes in an eddy turnover time.

take turns dominating the temperature and poloidal velocity. These functional forms are similar to those found in Paper II for $R_s = 10^4$, $Pr = 10$. The poloidal velocity smoothly decreases from its maximum value at the origin to zero at the surface, with no nodes and with no trace of the $r = 0.3$ shell that marks the boundary of the internal heat source. In contrast, the functional form of the high l components of the poloidal velocity does change in time. The number of nodes in the poloidal velocity (which corresponds physically to the number of radially stacked convective eddies) increases with l and is ~ 4 for $l = 12$. The functional form of the $l = 1$ temperature shown in Figure 12 is doubly peaked, with a small peak near the boundary layer and a large peak at $r = 0.3$. The toroidal component of the velocity is plotted as $r^3 \psi_{R,1,0}$ in Figure 14 and is the angular momentum about the z -axis. Conservation of angular momentum requires that

$$\int_0^1 r^3 \psi_{R,1,0} dr = 0$$

for all time. The functional form of $r^3 \psi_{R,1,0}$ changes dramatically in one eddy turnover time. No rotation law can be ascribed to the flow because for all values of radius the time-averaged value of $r^3 \psi_{R,1,0}$ is zero. It is important to realize that this numerical result does not immediately follow from angular momentum conservation.

b) Intermittency

In examining the intermittency of the flow (i.e., the fact that the turbulence is not uniform for all space and time) it is customary to calculate the departure of the inertial range spectrum from the Kolmogorov $k^{-5/3}$ law, the flatness of the velocity distribution (kurtosis), and the high-order correlations of the different components of the velocity (structure functions). However, because of our limited spatial resolution, our numerically computed values are not quantitatively useful measures of the spatial intermittency. Siggia and Patterson (1978), working in a finely resolved Cartesian geometry, have calculated the spatial intermittency for an isothermal fluid that is driven not by buoyancy but by a large-scale, isotropic, permanently frozen-in velocity field. Instead of trying to reproduce their results in a sphere of convecting fluid, we present some quantitative measures of the time dependence of the intermittency. Figure 15 shows the variations in the convective flux, $[F_{\text{con}} - \langle\langle F_{\text{con}} \rangle\rangle] / \langle\langle F_{\text{con}} \rangle\rangle$, at $r = 0.5$ as a function of time. The dashed line corresponds to the variance, σ (calculated over 30 eddy turnover times), in units of $\langle\langle F_{\text{con}} \rangle\rangle$. Each time step plotted in Figure 15 is 3×10^{-7} units of dimensionless time. Each 1000 time steps is roughly one eddy

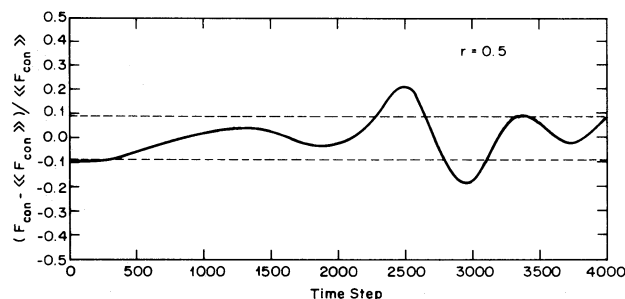


FIG. 15.—The deviations of the convective flux from its time-averaged value $\langle\langle F_{\text{con}} \rangle\rangle$ as a function of time. The figure is computed at $r = 0.5$. The dashed lines are the variance, σ , of F_{con} in units of $\langle\langle F_{\text{con}} \rangle\rangle$. One eddy-turnover time is approximately equal to 1000 time steps.

turnover time. There is a burst in the convective flux near iteration number 2500. The fluctuation is of order 2σ and lasts for approximately one-half of an eddy turnover time. We have found four bursts similar to the one illustrated in Figure 15 while examining the convective flux for 30 eddy turnover times. The bursts in flux are associated with bursts in the kinetic and thermal energy spectra. The fluctuations in these spectra at $r = 0.5$ for odd value of l are shown in Figures 16 and 17 as a function of time. The dashed lines correspond to the variances of the energy. All bursts are of order 2σ and clearly cascade from the low to the high wavenumber modes. It takes roughly one-half of an eddy turnover time for the kinetic energy bursts to cascade from $l = 1$ to $l = 9$. The thermal energy cascade is slower, and it requires nearly one turnover time to cascade from $l = 1$ to $l = 9$. Each of the bursts is initially positive, then followed by damped oscillations. We have never observed a burst that is initially negative.

The fluctuations seem to arise from a secondary convective instability in the boundary-layer where the local Rayleigh number is never far from its critical value. The fluctuations propagate inward from the surface via the $l = 1$ modes, then cascade downward in wavelength. The fluctuations in the high wavenumber modes at $r = 0.5$ are not caused by high wavenumber fluctuations that travel directly from the boundary layer to $r = 0.5$, but are due to energy cascading from low to high wavenumbers at $r = 0.5$. To prove this, we have shown that for $l > 2$, the 2σ fluctuation in $TE(l, r = 0.5)$ occurs just after a strong fluctuation in

$$\sum_{\gamma, m} (\mathbf{v} \cdot \nabla T_{<})_{\gamma, l, m} T_{\gamma, l, m}$$

at $r = 0.5$. The latter fluctuation is indicative of a cascade from big to small modes. Furthermore, we repeated the calculation starting with iteration number 1300 as the initial data. In the repeated calculation we artificially set the $(\mathbf{v} \cdot \nabla T_{<})_{\gamma, l, m}$ contribution to $(\mathbf{v} \cdot \nabla T)_{\gamma, l, m}$ equal to zero. We then found that there was still a 2σ fluctuation in the $l = 1$ modes, but that the fluctuations in $KE(l, r = 0.5)$ for $l > 2$ were less than 1σ . This proves that the $l > 2$ fluctuations are caused by energy cascading from big to small modes via the $(\mathbf{v} \cdot \nabla T_{<})$ terms.

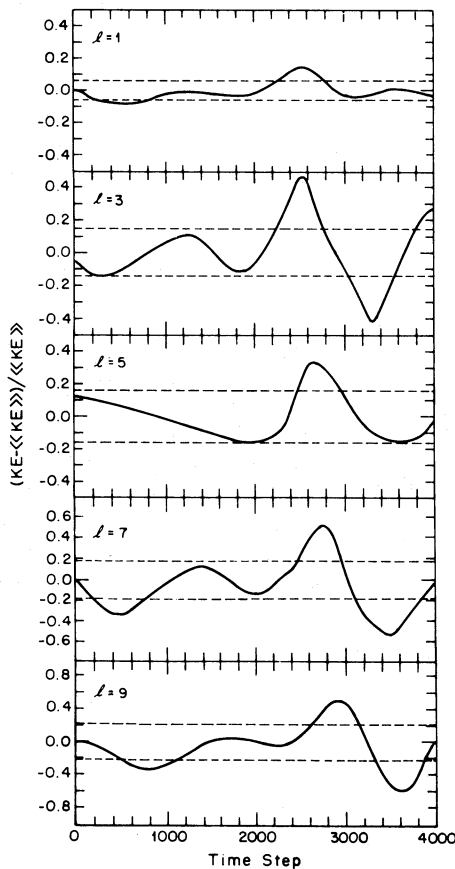


FIG. 16

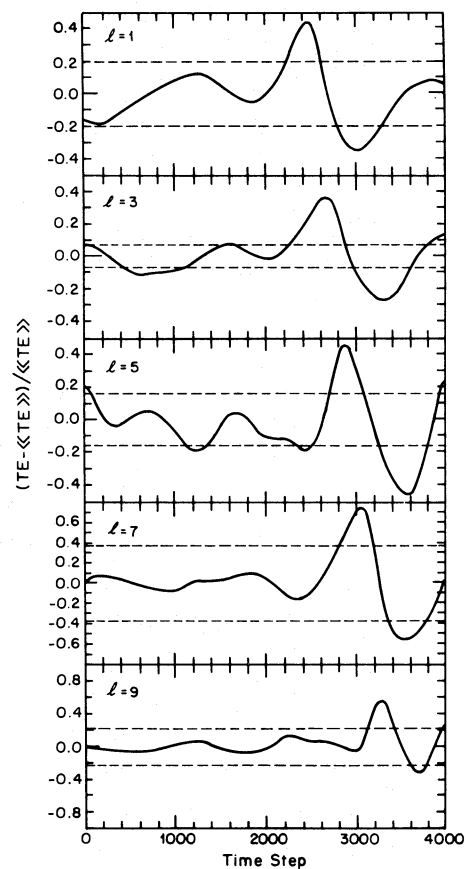


FIG. 17

FIG. 16.—The same as Fig. 15 for kinetic energy. The burst in energy requires \sim half a turnover time to cascade from $l = 1$ to $l = 9$.

FIG. 17.—The same as Fig. 16 for the fluctuating thermal energy. The thermal energy cascade is slower than the kinetic energy cascade and requires nearly one full turnover time to cascade from $l = 1$ to $l = 9$.

V. CONCLUSIONS

We have been able to show that for sufficiently high Rayleigh numbers, a convecting fluid develops an inertial subrange (in which the flow is nearly isotropic and the energy spectrum $\approx k^{-5/3}$) and that part of this subrange can be resolved numerically. The anisotropy of the flow and fine structure of the energy spectra are confined to the longest wavelengths. The effects of the unresolvable part of the inertial subrange can be modeled with an eddy viscosity and diffusivity that are functions only of the local values of the velocity field. The amplitudes of the eddy viscosities and diffusivity can be determined by requiring that the computed solutions be independent of the limit of spatial resolution. This method of computing the eddy viscosity fails to work if the limit of spatial resolution does not lie in the inertial subrange. Because we need a well-defined inertial range, our method of computing eddy viscosities requires that the Reynolds and Péclet numbers be large. We have found that the poloidal and toroidal parts of the eddy viscosity are nearly equal and that the eddy diffusivity is $\sim 10\%$ greater than the poloidal part of the eddy viscosity.

The important results obtained with these eddy viscosities with a Rayleigh number of 10^{10} and a Prandtl number of unity can be summarized as follows:

1. The fluid motions are time-dependent and have no spatial symmetries.
2. An examination of the kinetic and fluctuating thermal energy spectra as a function of wavelength as well as a study of the manner in which energy cascades in and out of each wavelength shows that there is an inertial range for $l \gtrsim 7$.
3. A computation of the ratio of the horizontal to the radial component of the kinetic energy shows that flow tends toward isotropy at $l \gtrsim 7$.
4. Except at the outer boundary layer, nearly all of the heat flux is carried by convection. There is overshooting at the center and near the boundary layer, causing the conductive flux to transport energy toward the center of the sphere.
5. The variance, σ (with respect to time), of the flux increases from the center to the surface of the sphere. At the surface, σ is $\sim 18\%$ of the total flux. The increase in σ near the surface is probably due to a secondary convective instability in the boundary layer.
6. The boundary-layer instabilities result in a series of intermittent bursts in flux and energy at all radii. The bursts in the energy cascade from the large, $l = 1$, to the small, $l = 9$, wavelengths require approximately one-half of an eddy turnover time for the kinetic energy and about one eddy turnover time for the thermal energy.

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