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STELLAR CONVECTION. I. MODAL EQUATIONS IN SPHERES AND SPHERICAL SHELLS*

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ABSTRACT

We derive a set of nonlinear, modal equations that is used to describe convection in self-gravitating spheres and spherical shells of Boussinesq fluids with internal heat sources that are functions of radius. We use a Galerkin method in which the (θ, ϕ)-dependence of the fluid is written as an infinite sum of eigenfunctions (planforms). The velocity field is written as a sum of two parts: the toroidal field which is generated from a pseudo-scalar and the poloidal field which is generated from a scalar. The nonlinear terms of the modal equations are written in terms of the scalar and pseudo-scalar, and the nonlinear three-eddy interactions are parametrized with a set of dimensionless B-numbers and C-numbers. We calculate these numbers for several sets of planforms. Special closed sets of planforms are used to describe highly symmetric convective flows. In particular, we find modes that have the rotational symmetries of the five regular solids and we compute their B-numbers and C-numbers. These modes are compared to the hexagonal modes that are observed in laboratory experiments. We show that all of the angular momentum of the fluid resides in one toroidal mode, and by studying this mode we examine the mechanism by which angular momentum is nonlinearly transferred among the different radial shells of fluid. We prove that any truncated set of modes rigorously conserves angular momentum provided that the boundaries of the fluid are stress-free.

Subject headings: convection — stars: interiors — stars: rotation — hydrodynamics

I. INTRODUCTION

This is the first in a series of four papers that numerically solve the hydrodynamic equations of convection. Unfortunately, presentations of numerical results often leave the reader with the nagging feeling that he is not quite sure of what was *really* computed or of the physical consequences of the numerical approximations that were made to make the equations tractable to computer solution. It is the purpose of this paper to show how we obtain a set of numerically solvable modal equations from the well-known hydrodynamic equations, to explore some of the analytic properties of these equations, and to determine the physical effects of our numerical approximations.

Although there have been many numerical studies of convection (see Spiegel 1971, 1972 for a review), none seem universally applicable to stars or numerically cheap enough to be employed in stellar evolution codes. Therefore, the star codes exclusively use a form of mixing-length theory. It is not the purpose of this paper to elucidate the pitfalls and virtues of mixing-length theory. However, it is clear that the mixing-length theory can never succeed if the convective motions are strongly coupled to other dynamical processes in the star such as rotation or pulsation, nor can the theory ever resolve the temporal or horizontal structure of the flow. In particular, mixing-length theory cannot determine the extent of overshoot that carries material from a convectively unstable region into the stable layers of a star (except by dimensional arguments about the energetics; cf. Shaviv and Salpeter 1973). The amount of mixing caused by this penetration can considerably alter stellar evolution by changing a star's main-sequence lifetime and its helium shell flashes (Paczyński 1977). The principal astrophysical calculation in this set of four papers is the quantitative determination of the stable, equilibrium flow and overshoot of the convective core of a zero-age main-sequence star.

We are motivated to use a modal analysis of convection because the convective velocity in a star can be physically thought of as an infinite sum of convective eddies and each eddy or mode as one component of the modal hydrodynamic equations. There is numerical evidence (Gough, Spiegel, and Toomre 1975; Toomre, Gough, and Spiegel 1977) that a truncated set of modal equations with only a few eddies (Galerkin method) can accurately reproduce the temperature gradient, transport properties, and time dependence of a convecting fluid in a laboratory. Although stellar convection has much higher velocities than most laboratory flows, there is some experimental evidence (Gollub and Swinney 1975; Whitehead and Parsons 1978) that the modal structure persists in convective flows when the Reynolds number becomes large. Furthermore, the observation of granulation and supergranulation on the solar surface presents the possibility that stellar convection is dominated by a finite number of well-defined modes. Latour and co-workers (Latour *et al.* 1976; Toomre *et al.* 1976) examined convection in the

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second ionization zone of an A star with a modal analysis that they limited to a single mode with a plane-parallel geometry. They found that the convective heat flux was several orders of magnitude greater than the one predicted by mixing-length theory and was accompanied by a large overshoot. By using several hundred modes in our calculation, we shall be able to test the hypothesis that only a few eddies are responsible for the heat transport in stellar convection.

In this first paper we derive the modal equations in spherical coordinates for a Boussinesq fluid (i.e., the depth is much less than pressure scale-height), paying particular attention to developing a notation that facilitates the numerical computation of a large number ($\sim$200) of nonlinearly coupled modes. Although the Boussinesq approximation has only limited usefulness in approximating a stellar fluid (Spiegel and Veronis 1960), it allows us to master the numerical techniques that are needed to find self-consistent, stable, equilibrium solutions to a truncated set of equations without the additional approximations and messier physics associated with compressible fluids.

Paper II is really an experimental paper. In it, we not only present solutions to the Boussinesq equations, but also report a large number of numerical experiments performed with these equations and their truncation. The greatest difficulty in using a modal representation for a velocity field is the compromise that is needed between the infinite number of modes that represent the velocity exactly and the finite number of modal equations that can be integrated efficiently. Our numerical experiments establish the best compromise. In order to portray a turbulent fluid with a modal analysis, we would at most need to include a complete set of eddies whose size is greater than the smallest physical length that appears in the problem—the Kolmogorov microscale (see Tennekees and Lumley 1972). On the other hand, if classical spectral theory is not accurate for high Reynolds number convective flows and if only a few eddies dominate the velocity fields, then we only need a much smaller set of highly specialized modes. Paper II tests numerically the validity of spectral theory for some specific examples of convective flow. To allow us to find solutions to the large number of modal equations, we introduce a new relaxation method which allows us to converge quickly to equilibrium solutions (at the expense of losing information about the temporal phase). The solutions are then tested for stability.

Armed with the knowledge of the modal structure of a self-gravitating sphere of Boussinesq fluid, we derive the modal equations for a self-gravitating sphere of ideal gas in Paper III. The spirit of the paper is to cast the compressible modal equations into a form that is analogous to the Boussinesq equations; we find an analog to the Rayleigh number, which is rigorously defined only for a Boussinesq fluid with a plane-parallel geometry and which determines the onset of convective instability.

Like Paper II, Paper IV is experimental. We examine convection in self-gravitating spheres of hydrogen with masses and luminosities that correspond to upper main-sequence stars. We compute several stable equilibrium flows in which the convective overshoot is present. In this fourth paper, the large eddies are studied with modal analysis and the small eddies with spectral theory. We match the solution at an intermediate wavelength and show that the complete solution is independent of the wavelength at which we join the two solutions together (as long as that wavelength lies in the inertial-convective subrange). The large eddies, which are described by modal analysis, are astrophysically interesting—they represent the overshoot, transport most of the heat, contain most of the kinetic energy, and have the dynamical interactions that determine the relative stability of different equilibrium solutions.

This paper, which by its nature is somewhat mathematical, presents the formalism that is used in the next three papers. Section II begins with the Boussinesq equations in spherical coordinates. In § III the velocity is written as the sum of its toroidal part, generated by a pseudo-scalar, and its poloidal part, generated by a scalar. The hierarchy of coupled modal equations is derived. The kinematics of the nonlinear interactions are parameterized in terms of C-numbers and B-numbers; Baker and Spiegel (1975) used a similar parameterization for the three-eddy interactions in a plane-parallel geometry. In § IV we derive the boundary conditions for spherical shells and spheres of fluid. In a sphere we do not require that the velocity vanish at the origin. In § V we select the complete set of modes that are used in the numerical calculations of Paper II. C-numbers and B-numbers are calculated in terms of the Wigner $3j$ symbols. We derive modes that have the same rotational symmetries of the five regular solids. These modes are in some ways similar to the hexagonal modes that are observed in laboratory convection (Christopherson 1940). The fundamental dodecahedral and hexahedral modes are of special interest because Busse (1975) has shown that they are the preferred modes of convection in marginally stable fluids. In § VI we examine the mechanics of angular momentum transfer among different shells of fluid and show how the total angular momentum is exactly conserved by an arbitrarily truncated set of modal equations. We solve analytically the equations for the case of solid-body rotation, and with this simple example we explicitly exhibit the mechanics of the nonlinear interactions of modes. In § VII we discuss our rationale for truncating the modal equations, and in the last section we compare this paper to other modal studies of convection.

II. HYDRODYNAMIC EQUATIONS OF MOTION

a) *Boussinesq Equations*

Four independent quantities that can be used to describe convection in a self-gravitating fluid are the pressure P; the density ρ; the temperature T; and the velocity field v. In this paper we shall only consider convection in a

Boussinesq fluid, i.e., one whose depth is much less than its pressure scale-height and in which there is no viscous dissipation of kinetic energy. We further restrict the fluid so that its heat capacity c_p, kinematic viscosity ν, and thermal diffusivity k are uniform throughout the fluid and independent of temperature and pressure. The four equations that we use to determine T, P, ρ, and $\boldsymbol{v}$ are the Boussinesq momentum equation:

$$\partial \boldsymbol{v}/\partial t = -(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nabla P/\rho_0 + (1 + \delta\rho/\rho_0)\boldsymbol{x} + \nu\nabla^2\boldsymbol{v}\,; \tag{2.1}$$

the Boussinesq energy equation:

$$\partial T/\partial t = -(\boldsymbol{v}\cdot\nabla)T + k\nabla^2 T + H/c_p\rho_0\,; \tag{2.2}$$

the equation of state:

$$\rho \equiv \rho_0 + \delta\rho = \rho_0[1 + \alpha(T_0 - T)]\,; \tag{2.3}$$

and the Boussinesq continuity equation which requires that the velocity field be identically solenoidal:

$$\nabla\cdot\boldsymbol{v} = 0\,. \tag{2.4}$$

In the preceding equations $H(r, \theta, \phi)$ is the energy generation rate per unit volume, α is the coefficient of thermal expansion, and $\boldsymbol{x}$ is the gravitational acceleration. (For a discussion of the Boussinesq equations, see Chandrasekhar 1961). Instead of supplying an additional equation for the energy generation rate, H, as a function of T and P (as is customary in stellar work), we treat H as a part of the initial conditions and hold it permanently fixed. Equations (2.1)–(2.4) are applicable to a rotating as well as a nonrotating fluid, if we choose our boundary conditions properly. We avoid the conventional techniques of examining a rotating fluid in a rotating frame with a fictional Coriolis force for reasons that will be discussed in § V.

As is customary, we write each dependent variable $f(r, \theta, \phi)$, as a sum of its mean value $\langle f(r)\rangle$ and its fluctuation $\tilde{f}(r, \theta, \phi)$. The mean part is defined

$$\langle f(r)\rangle \equiv \int_{4\pi} d\Omega f(r, \theta, \phi)/4\pi\,; \tag{2.5}$$

and the fluctuation is defined.

$$\tilde{f}(r, \theta, \phi) \equiv f(r, \theta, \phi) - \langle f(r)\rangle\,. \tag{2.6}$$

To order $\delta\rho/\rho_0$ (the accuracy of the Boussinesq equations), the fluctuating part of the gravitational term in the momentum equation is

$$(1 + \delta\tilde{\rho}/\rho_0)\boldsymbol{x} = -\nabla\langle\Phi\rangle\tilde{\rho}/\rho_0 - \nabla\tilde{\Phi}\,, \tag{2.7}$$

where Φ is the gravitational potential. Φ is determined from Poisson's equation and $\langle\Phi\rangle$ is

$$\nabla\langle\Phi(r)\rangle \equiv -\nabla\left[G\int dr' d\Omega' r'^2 \langle 1/|r - r'|\rangle \rho(r', \theta', \varphi')\right] \tag{2.8}$$

$$= (4\pi/3)G\rho_0 r^{-2}(r^3 - r_1^3)\hat{e}_r[1 + O(\delta\rho/\rho_0)]\,, \tag{2.9}$$

where the shell, $r = r_1$, is the inner boundary of the fluid.

We obtain the equation for the fluctuating part of the potential and pressure by taking the fluctuating part of the divergence of the momentum equation and by using the equation of state, yielding

$$\nabla^2(\tilde{P}/\rho_0 + \tilde{\Phi}) = (4\pi/3)G\rho_0\{3\tilde{T} + r[1 - (r_1/r)^3]\partial\tilde{T}/\partial r\} - \nabla\cdot[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}] + \langle\nabla\cdot[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}]\rangle\,. \tag{2.10}$$

The term that is nonlinear in the velocity is identified as the "turbulent pressure," and it is due to the nonlinear interactions among three advective eddies. Using Poisson's equation with the equation of state, we obtain

$$\nabla^2\tilde{\Phi} = -4\pi G\rho_0\alpha\tilde{T}\,. \tag{2.11}$$

The fluctuating part of the Boussinesq energy equation gives the equation for the time-development of $\tilde{T}$:

$$\partial\tilde{T}/\partial t = k\nabla^2\tilde{T} - \tilde{v}_r\partial\langle T\rangle/\partial r - \boldsymbol{v}\cdot\nabla T + r^{-2}\partial\langle r^2 v_r T\rangle/\partial r + \tilde{H}/c_p\rho_0\,. \tag{2.12}$$

The nonlinear term $(\boldsymbol{v}\cdot\nabla)T$ is the "turbulent conductivity" and represents the thermal energy cascade. The linear radial advection of thermal energy is given by $\tilde{v}_r\partial\langle T\rangle/\partial r$.

In a Boussinesq fluid, the only mean scalar quantity that must be determined is the mean temperature gradient. This quantity is readily obtained by taking the mean part of equation (2.2) and integrating with respect to r:

$$\partial\langle T(r)\rangle/\partial r = \langle\tilde{v}_r\tilde{T}\rangle/k - \mathscr{L}(r)/4\pi\rho_0 c_p k r^2 + r^{-2}\partial/\partial t\int_{r_1}^{r} dr' r'^2\langle T(r')\rangle/k\,, \tag{2.13}$$

where $\mathscr{L}(r)$ is the total rate of energy production (luminosity) interior to the radius r,

$$\mathscr{L}(r) \equiv 4\pi \int_{r_1}^{r} \langle H \rangle r'^2 \, dr' , \tag{2.14}$$

and where r_1 is the radius of the inner boundary. Equation (2.13) shows that the time-independent, conductive temperature gradient is proportional to $-\mathscr{L}(r)/r^2$.

b) Nondimensionalization and Dimensionless Constants

Throughout the remainder of this paper (unless otherwise noted) all variables are nondimensional. The unit of length is $(r_2 - r_1)$, where r_2 is the outer radius of the sphere or spherical shell of fluid and r_1 is the inner radius, $(r_2 - r_1)^2/k$ is the unit of time and is the characteristic time it takes to conduct energy across the depth of the fluid, $\rho_0(r_2 - r_1)^3$ is the unit of mass, and the unit of temperature is $\mathscr{L}(r_2)/4\pi\rho_0 c_p(r_2 - r_1)k$. With these units, three dimensionless numbers appear in the Boussinesq equations: the Prandtl number, $\mathrm{Pr} \equiv \nu/k$; the "spherical Rayleigh number," $\mathrm{Rs} \equiv \alpha G(r_2 - r_1)^3 \mathscr{L}(r_2)/3k^2\nu c_p$, which acts in the spherical equations in the same manner in which the Rayleigh number $\mathrm{Ra} \equiv \alpha d^3[T(r_2) - T(r_1)]g/k\nu$ does in a plane-parallel geometry; and a third dimensionless number, $\delta \equiv r_1/(r_2 - r_1)$. Ra can be identified with Rs if we identify d with $(r_2 - r_1)$, g with $4\pi G\rho_0(r_2 - r_1)/3$, and $[T(r_2) - T(r_1)]$ with $\mathscr{L}(r_2)/4\pi\rho_0 c_p k(r_2 - r_1)$. $\mathscr{L}(r_2)/4\pi\rho_0 c_p k(r_2 - r_1)$ is equal to the dimensionless number $(r_1 - r_2)^2/r_1 r_2$ multiplied by the temperature difference $[T^c(r_2) - T^c(r_1)]$, where $T^c(r_1)$ and $T^c(r_2)$ are the temperatures at the inner and outer boundaries of the fluid in conductive equilibrium if all of the energy production $\mathscr{L}(r_2)$ were located at r_1. The dimensionless radial coordinate, r_{dim}, is defined

$$r_{\mathrm{dim}} \equiv (r - r_1)/(r_2 - r_1) . \tag{2.15}$$

We subsequently drop the subscript "dim" from the dimensionless radial coordinate.

III. MODAL EQUATIONS

a) Planforms, B-Numbers, and C-Numbers

In the modal analyses of the hydrodynamic equations of motion we represent each fluctuating quantity as a sum of coefficients that are functions only of radius multiplied by a set of basis functions that are functions only of θ and φ. In the modal treatment of convection in a plane-parallel geometry, the set of basis functions is chosen to be a complete orthonormal set of eigenfunctions of the horizontal Laplacian, $(\partial^2/\partial x^2 + \partial^2/\partial y^2)$, that satisfy an appropriate set of horizontal boundary conditions. For the spherical geometry we choose the basis functions to be eigenfunctions of the horizontal Laplacian,

$$\left(\nabla_s^2 \equiv \nabla^2 - \frac{1}{r}\frac{\partial^2}{\partial r^2} r \right) ,$$

and the only boundary condition we impose is single-valuedness in θ, φ. We define the horizontal wavelength, γ, of a spherical planform, f:

$$\nabla_s^2 f(r, \theta, \varphi) = -(4\pi^2/\gamma^2) f(r, \theta, \varphi) . \tag{3.1}$$

One possible choice of basis functions is the spherical harmonics $Y^{l,m}$, and each harmonic has wavelength $2\pi[l(l+1)]^{-1/2}$. Let us consider a complete set of basis functions $\{h^{l,\alpha}(\theta, \varphi)\}$, where l is a positive integer, $\langle h^{l,\alpha} \rangle = 0$, $\langle h^{l,\alpha} h^{l',\alpha'} \rangle = \delta_{l,l'}\delta_{\alpha,\alpha'}$, and $\nabla_s^2 h^{l,\alpha} = -l(l+1)h^{l,\alpha}$. We adopt the notation that any function written with subscripts l, α is defined

$$f_{l,\alpha} \equiv \langle \tilde{f}(r, \theta, \varphi) h^{l,\alpha} \rangle . \tag{3.2}$$

To facilitate writing the modal equations we extend the B-number and C-number notation that was used by Baker and Spiegel (1975) for their plane-parallel geometries. We define the C-number and B-number:

$$C(h^{l,\alpha}, h^{l',\alpha'}, h^{l'',\alpha''}) = \langle h^{l,\alpha}, h^{l',\alpha'}, h^{l'',\alpha''} \rangle , \tag{3.3}$$

$$B(h^{l,\alpha}, h^{l',\alpha'}, h^{l'',\alpha''}) = \langle h^{l,\alpha} \hat{e}_r \cdot (\nabla_s h^{l',\alpha'} \times \nabla_s h^{l'',\alpha''}) \rangle , \tag{3.4}$$

where ∇_s is the horizontal gradient operator and is defined by its action on the function η:

$$\nabla_s n \equiv r\nabla\eta - r\partial\eta/\partial r \hat{e}_r .$$

The B and C numbers are useful because they encapsulate all of the multiplicative properties of $\{h^{l,\alpha}\}$:

$$h^{l',\alpha'} \cdot h^{l'',\alpha''} = \sum_{l,\alpha} C(h^{l,\alpha}, h^{l',\alpha'}, h^{l'',\alpha''}) h^{l,\alpha}$$

and

$$\hat{e}_r \cdot (\nabla_s h^{l',\alpha'} \times \nabla_s h^{l'',\alpha''}) = \sum_{l,\alpha} B(h^{l,\alpha}, h^{l',\alpha'}, h^{l'',\alpha''}) h^{l,\alpha} \, .$$

C is obviously invariant under all interchanges of its arguments. B is invariant under cyclic permutations but changes sign if any two of its arguments are interchanged. Because of its antisymmetry, a B number with two or more identical arguments is zero. Furthermore, if l, l', or l'' is equal to zero, then B also equals zero. The symmetry properties of B can be proved by writing the horizontal averaging as an integral and integrating by parts. By expanding the basis functions $\{h^{l,\alpha}\}$ in terms of the spherical harmonics it is obvious that $C(h^{l,\alpha}, h^{l',\alpha'}, h^{l'',\alpha''})$ is zero unless the sum of l, l', and l'' is an even integer (parity rule) and unless $|l' - l''| \leq l \leq l' + l''$ (triangle inequality). There are also selection rules for the α's, but we defer that discussion until § V. The l's in the B number must also obey a parity rule and triangle inequality: for B to be nonzero, the sum of l, l', and l'' must be an odd integer and $|l' - l''| + 1 \leq l \leq l' + l'' - 1$.

b) *Representation of the Poloidal and Toroidal Components of the Velocity*

In a model analysis, it is awkward to deal with vector fields. Therefore, we write the solenoidal velocity as a sum of its poloidal and toroidal parts:

$$\boldsymbol{v} \equiv \boldsymbol{v}_P + \boldsymbol{v}_T \, , \tag{3.5}$$

where $\boldsymbol{v}_P$ *is generated* by the scalar ω

$$\boldsymbol{v}_P \equiv \nabla[\partial(r\omega)/\partial r] - (r\nabla^2\omega)\hat{e}_r \, , \tag{3.6}$$

and $\boldsymbol{v}_T$ *is generated* by the pseudo-scalar ψ

$$\boldsymbol{v}_T \equiv r\nabla \times (\psi\hat{e}_r) \, . \tag{3.7}$$

The toroidal part, $\boldsymbol{v}_T$, is divergence-free and has no radial component; the poloidal part, $\boldsymbol{v}_P$, is also divergence-free, and its curl has no radial component. (In fact, $\nabla \times \boldsymbol{v}_P$ is toroidal and $\nabla \times \boldsymbol{v}_T$ is poloidal.) Our definitions of the generating scalars, ω and ψ, differ somewhat from convention (Chandrasekhar 1961), but equations (3.5)–(3.7) are completely general and our definitions have the advantage that if ω generates $\boldsymbol{v}_P$, then $\nabla^2\omega$ generates $\nabla^2\boldsymbol{v}_P$ and if ψ generates $\boldsymbol{v}_T$, then $\nabla^2\psi$ generates $\nabla^2\boldsymbol{v}_T$. That is, given (3.6) and (3.7), then

$$\nabla^2\boldsymbol{v}_P = \nabla[\partial(r\nabla^2\omega)/\partial r] - (r\nabla^2\nabla^2\omega)\hat{e}_r \, , \tag{3.8}$$

$$\nabla^2\boldsymbol{v}_T = r\nabla \times [(\nabla^2\psi)\hat{e}_r] \, . \tag{3.9}$$

Instead of discussing the modal equation for the three dependent components of the velocity field, we shall find modal equations for the two independent quantities $\partial\omega/\partial t$ and $\partial\psi/\partial t$.

By taking the mean part of equation (2.4), we see that the mean part of the radial component of the velocity vanishes

$$\langle v_r \rangle = 0 \tag{3.10}$$

or

$$\langle \omega \rangle = 0 \, . \tag{3.11}$$

It is not true that $\langle v_\theta \rangle$ or $\langle v_\varphi \rangle$ must vanish, but we always have the gauge freedom to require that

$$\langle \psi \rangle = 0 \, . \tag{3.12}$$

In the modal analysis, we will need the fact that $\psi = \tilde{\psi}$ and $\omega = \tilde{\omega}$.

c) *Modal Equations for ω, ψ, P, T, and Φ*

Casting the equations of motion into modal form requires us to find the equations for $T_{l,\alpha}$, $P_{l,\alpha}$, $\Phi_{l,\alpha}$, $\omega_{l,\alpha}$, $\psi_{l,\alpha}$, and $\partial\langle T\rangle/\partial r$.

From (2.12) the equation for $T_{l,\alpha}$ is

$$\begin{aligned}\partial T_{l,\alpha}/\partial t = {} & [1/(r+\delta)\partial^2/\partial r^2(r+\delta) - l(l+1)/(r+\delta)^2]T_{l,\alpha} - l(l+1)(\partial\langle T\rangle/\partial r)\omega_{l,\alpha}/(r+\delta) + H_{l,\alpha} \\ & - (r+\delta)^{-2} \sum_{l',\alpha';\, l'',\alpha''} \{T_{l',\alpha'}\partial[(r+\delta)\omega_{l'',\alpha''}]/\partial r\}\tfrac{1}{2}[l'(l'+1) + l''(l''+1) - l(l+1)] \\ & + [l''(l''+1)\omega_{l'',\alpha''}\partial T_{l',\alpha'}/\partial r](r+\delta)\}C(h^{l,\alpha}, h^{l',\alpha'}, h^{l'',\alpha''}) \\ & - (r+\delta)^{-1} \sum_{l',\alpha';\, l'',\alpha''} \psi_{l'',\alpha''}T_{l',\alpha'}B(h^{l,\alpha}, h^{l',\alpha'}, h^{l'',\alpha''}) \, .\end{aligned} \tag{3.13}$$

The sums in equation (3.13) are over all l', α' and l'', α'' for which the C numbers and B numbers are nonzero.

It is physically useful to think of each mode, $T_{l,\alpha}$, as a convective eddy. Equation (3.13) then tells us that the temperature of each eddy is affected by (1) the thermal diffusion of itself, (2) advection of heat by the poloidal component of the eddy's velocity, (3) the heat generated in that eddy, and (4) the "turbulent" heat conduction—the thermal exchange of energies among itself and all other eddies (l', α') and (l'', α'') that have a nonzero value of $C(h^{l,\alpha}, h^{l',\alpha'}, h^{l'',\alpha''})$ or $B(h^{l,\alpha}, h^{l',\alpha'}, h^{l'',\alpha''})$.

The modal equation for the poloidal generator of the velocity is obtained from the radial part of the Boussinesq momentum equation (2.1):

$$\partial\omega_{l,\alpha}/\partial t = -[l(l+1)]^{-1}[\![-\mathrm{PrRs}(r+\delta)^2\{1-[\delta/(r+\delta)]^3\}T_{l,\alpha} + (r+\delta)\partial(P_{l,\alpha}/\rho_0 + \Phi_{l,\alpha})/\partial r]\!]$$
$$+ \Pr\{(r+\delta)^{-1}\partial^2[(r+\delta)\omega_{l,\alpha}]/\partial r^2 - l(l+1)\omega_{l,\alpha}/(r+\delta)^2\} - [l(l+1)]^{-1}\{(r+\delta)\hat{e}_r\cdot[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}]\}_{l,\alpha}\,. \quad (3.14)$$

The nonlinear term, $\{\hat{e}_r\cdot[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}]\}_{l,\alpha}$, is due to the "turbulent viscosity" and quantitively describes the cascade of energy into and out of the (l, α) eddy. In terms of ω and ψ, we find

$$\{(r+\delta)\hat{e}_r\cdot[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}]\}_{l,\alpha} = \sum_{l',\alpha';\,l'',\alpha''} (\omega_{l',\alpha'}\{\partial[\omega_{l'',\alpha''}/(r+\delta)]/\partial r\}l'l''(l'+1)(l''+1) - \tfrac{1}{2}[l(l+1) - l'(l'+1) - l''(l''+1)]$$
$$\times [\![\{\partial[(r+\delta)\omega_{l',\alpha'}]/\partial r\}(r+\delta)^{-2}\{l''(l''+1)\omega_{l'',\alpha''} - \partial[(r+\delta)\omega_{l'',\alpha''}]/\partial r\} - \psi_{l',\alpha'}\psi_{l'',\alpha''}]\!])$$
$$\times C(h^{l,\alpha}, h^{l',\alpha'}, h^{l'',\alpha''}) + (r+\delta)^{-1}\sum_{l',\alpha';\,l'',\alpha''} \psi_{l',\alpha'}\{-l''(l''+1)\omega_{l'',\alpha''} + 2[(r+\delta)\omega_{l'',\alpha''}]/\partial r\}$$
$$\times B(h^{l,\alpha}, h^{l',\alpha'}, h^{l'',\alpha''})\,. \quad (3.15)$$

By taking the radial component of the curl of the momentum equation, we obtain the modal equation for the toroidal generator:

$$\partial\psi_{l,\alpha}/\partial t = \Pr\{(r+\delta)^{-1}\partial^2[(r+\delta)\psi_{l,\alpha}]/\partial^2 r - l(l+1)\psi_{l,\alpha}/(r+\delta)^2\}$$
$$- [l(l+1)]^{-1}\{(r+\delta)\hat{e}_r\cdot\nabla\times[\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\}_{l,\alpha}\,. \quad (3.16)$$

Notice that the poloidal part of the velocity is driven by the fluctuations in the pressure, temperature, and gravitational potential, whereas the toroidal part of the velocity cannot feel these fluctuations directly and is only affected by viscosity and nonlinear interactions. We also note that the viscous drag of the poloidal (toroidal) part of the velocity only affects the poloidal (toroidal) velocity.

The nonlinear term is due to turbulent viscosity and, written in terms of ψ and ω, is:

$$\{(r+\delta)\hat{e}_r\cdot\nabla\times[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}]\}_{l,\alpha} = -\tfrac{1}{2}(r+\delta)^{-2}\sum_{l',\alpha';\,l'',\alpha''}[l'(l'+1) - l''(l''+1) - l(l+1)]$$
$$\times l'(l'+1)\{\omega_{l',\alpha'}\partial[(r+\delta)\psi_{l'',\alpha''}]/\partial r - \psi_{l',\alpha'}\partial[(r+\delta)\omega_{l'',\alpha''}]/\partial r\}C(h^{l,\alpha}, h^{l',\alpha'}, h^{l'',\alpha''})$$
$$- (r+\delta)^{-2}\sum_{l',\alpha';\,l'',\alpha''} l''(l''+1)\{\partial^2[(r+\delta)\omega_{l',\alpha'}]/\partial r^2\omega_{l'',\alpha''} + (r+\delta)\psi_{l',\alpha'}\psi_{l'',\alpha''}\}$$
$$\times B(h^{l,\alpha}, h^{l',\alpha'}, h^{l'',\alpha''})\,. \quad (3.17)$$

Notice that the nonlinear toroidal-toroidal contribution of the $h^{l',\alpha'}$ and $h^{l'',\alpha''}$ modes to $\partial\psi_{l,\alpha}/\partial t$ is

$$[l(l+1)(r+\delta)]^{-1}\psi_{l',\alpha'}\psi_{l'',\alpha''}[l''(l''+1)B(h^{l,\alpha}, h^{l',\alpha'}, h^{l'',\alpha''}) + l'(l'+1)B(h^{l,\alpha}, h^{l'',\alpha''}, h^{l',\alpha'})]\,,$$

and is equal to zero if $l' = l''$ regardless of α' and α'' due to the antisymmetry of the B number. The modal equation for the pressure is obtained from equation (2.10):

$$\left[(r+\delta)^{-1}\frac{\partial^2}{\partial r^2}(r+\delta) - l(l+1)(r+\delta)^{-2}\right]P_{l,\alpha}$$
$$= \mathrm{Rs}\,\Pr[\![6T_{l,\alpha} + (r+\delta)\{1-[\delta/(r+\delta)]^3\}\partial T_{l,\alpha}/\partial r]\!] - (r+\delta)^{-2}\partial\{(r+\delta)[(r+\delta)\hat{e}_r\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{v})]_{l,\alpha}\}/\partial r$$
$$+ \tfrac{1}{2}(r+\delta)^{-3}\sum_{l',\alpha';\,l'',\alpha''}[\![[l(l+1) - l'(l'+1) + l''(l''+1)]l'(l'+1)\omega_{l',\alpha'}\partial^2[(r+\delta)\omega_{l'',\alpha''}]/\partial r^2$$
$$- \tfrac{1}{2}(r+\delta)^{-1}l(l+1)[l(l+1) - 2l'(l'+1)]\{\partial[(r+\delta)\omega_{l',\alpha'}]/\partial r\}\{\partial[(r+\delta)\omega_{l'',\alpha''}]/\partial r\}$$
$$- \tfrac{1}{2}(r+\delta)\psi_{l',\alpha'}\psi_{l'',\alpha''}\{l(l+1)[l(l+1)$$
$$- 4l''(l''+1)] - 2l'(l'+1)l''(l''+1) + 2l'l'(l'+1)^2\}]\!]C(h^{l,\alpha}, h^{l',\alpha'}, h^{l'',\alpha''})$$
$$- (r+\delta)^{-3}\sum_{l',\alpha';\,l'',\alpha''}\{[l(l+1) - l'(l'+1)]\psi_{l',\alpha'}\partial[(r+\delta)\omega_{l'',\alpha''}]/\partial r$$
$$+ l'(l'+1)\omega_{l',\alpha'}\partial[(r+\delta)\psi_{l'',\alpha''}]/\partial r\}B(h^{l,\alpha}, h^{l',\alpha'}, h^{l'',\alpha''})\,. \quad (3.18)$$

The (l, α)-component of the gravitational potential is given by equation (2.11):

$$\left[(r+\delta)^{-1}\frac{\partial^2}{\partial r^2}(r+\delta) - l(l+1)/(r+\delta)^2\right]\Phi_{l,\alpha} = -3\,\mathrm{Pr}\,\mathrm{Rs}\,T_{l,\alpha}\,. \tag{3.19}$$

Using equation (2.13), the mean temperature gradient in terms of the modes becomes

$$\partial\langle T\rangle/\partial r = -\mathscr{L}(r)/(r+\delta)^2 + (r+\delta)^{-2}\partial/\partial t\int_0^r dr'(r'+\delta)^2\langle T\rangle + 1/(r+\delta)\sum_{l,\alpha} T_{l,\alpha}\omega_{l,\alpha}l(l+1)\,. \tag{3.20}$$

Notice that the mean temperature gradient is not directly affected by the toroidal velocity because the toroidal velocity has no radial component.

The kinetic energy, KE, of the fluid is easily expressed in terms of modes:

$$\mathrm{KE} \equiv \tfrac{1}{2}\int \boldsymbol{v}\cdot\boldsymbol{v}d^3r \equiv \sum_l \mathrm{KE}_l^P + \sum_l \mathrm{KE}_l^T\,, \tag{3.21}$$

where KE_l^P is the total amount of kinetic energy associated with the poloidal velocity field with wavelength $2\pi[l(l+1)]^{-1/2}$ and KE_l^T is associated with the toroidal field with that wavelength:

$$\mathrm{KE}_l^P = 2\pi l(l+1)\sum_\alpha \int_0^1 dr\, l(l+1)\omega_{l,\alpha}^2 + \{\partial[(r+\delta)\omega_{l,\alpha}]/\partial r\}^2\,, \tag{3.22}$$

$$\mathrm{KE}_l^T = 2\pi l(l+1)\sum_\alpha \int_0^1 dr\,(r+\delta)^2\psi_{l,\alpha}^2\,. \tag{3.23}$$

IV. BOUNDARY CONDITIONS

There are two distinct geometries that we consider in this paper: the spherical shell and the sphere.

a) The Spherical Shell ($\delta \neq 0$)

We impose the conditions that $\delta \neq 0$ and that the boundaries at $r = 0$ and $r = 1$ are impermeable, $v_r(0) = v_r(1) = 0$, and stress-free. In terms of modes, impermeability requires that

$$\omega_{l,\alpha}(0) = \omega_{l,\alpha}(1) = 0 \tag{4.1}$$

for all l, α. For the stress to vanish with impermeable boundaries, it is necessary and sufficient that

$$\partial^2\omega_{l,\alpha}/\partial r^2|_{r=0} = \partial^2\omega_{l,\alpha}/\partial r^2|_{r=1} = 0 \qquad \text{for all } l,\alpha \tag{4.2}$$

and

$$\partial[\psi_{l,\alpha}/(r+\delta)]/\partial r|_{r=0} = \partial[\psi_{l,\alpha}/(r+\delta)]/\partial r|_{r=1} \qquad \text{for all } l,\alpha\,. \tag{4.3}$$

The boundary conditions on the fluctuating part of the temperature are set by making the surfaces at $r = 0$ and $r = 1$ infinitely conducting; this forces $\partial\tilde{T}/\partial\theta = \partial\tilde{T}/\partial\varphi = 0$ at the boundaries, or

$$T_{l,\alpha}(0) = T_{l,\alpha}(1) = 0 \qquad \text{for all } l,\alpha\,. \tag{4.4}$$

Furthermore, if $\tilde{H}(r) = 0$ at $r = 0$ (or $r = 1$), then $\partial^2[(r+\delta)T_{l,\alpha}]/\partial r^2 = 0$ at $r = 0$ (or $r = 1$). By requiring that there are no sources or sinks of heat interior to the boundary at $r = 0$, equation (3.20) shows that

$$\partial\langle T\rangle/\partial r|_{r=0} = 0\,. \tag{4.5}$$

At the outer boundary we have

$$\partial\langle T\rangle/\partial r|_{r=1} = -\mathscr{L}(1)/(\delta+1)^2 + (\delta+1)^{-2}\frac{\partial}{\partial t}\int_0^1 dr'(r'+\delta)^2\langle T\rangle\,. \tag{4.6}$$

In the steady state, the mean temperature gradient at the outer surface is the gradient that is required to conduct the total luminosity, $\mathscr{L}(1)$, since the convective flux is zero. The mean temperature $\langle T\rangle$ never appears in the Boussinesq equations without a time or spatial derivative and is therefore determined only to to an additive constant. Let us fix $\langle T(0)\rangle = T_1$ and then determine $\langle T(1)\rangle$ from the equations of motion. A convenient dimensionless number which can be used as a measure of the efficiency of convection is $N \equiv [\langle T(1)\rangle - \langle T(1)\rangle^c]$, where $\langle T(1)\rangle^c$ is the temperature at $r = 1$ when the fluid is in conductive equilibrium:[1]

$$N = \int_0^1 dr\left[(r + \delta)^{-1} \sum_{l,\alpha} l(l + 1)T_{l,\alpha}\omega_{l,\alpha}\right] + \partial/\partial t \int_0^1 dr(r + \delta)^{-2} \int_0^r dr'(r' + \delta)^2\langle T\rangle\,. \tag{4.7}$$

The boundary conditions on the gravitational potential are uniquely determined by using the Neumann Green's function consistent with no mass interior to $r = 0$ or exterior to $r = 1$; the boundary conditions on $\tilde{P}$ are uniquely determined from all of the previously mentioned boundary conditions and the equations of motion. In practice, we are not interested in the values of $P_{l,\alpha}$ and $\Phi_{l,\alpha}$, but only in the value of $(P + \Phi)_{l,\alpha}$, since it is only the latter expression that appears in the modal Boussinesq equations. $(P + \Phi)_{l,\alpha}$ is determined from Poisson's equation:

$$\nabla^2[(P + \Phi)_{l,\alpha}h^{l,\alpha}] = \chi_{l,\alpha}h^{l,\alpha}\,, \tag{4.8}$$

where $\chi_{l,\alpha}$ is the right-hand side of equation (3.18). The numerically practical method of solving equation (4.18) is to use the *infinite* domain Green's function of the Laplacian, expand $1/|x - x'|$ in terms of $\{h^{l,\alpha}\}$, and include constants of integration, $a_{l,\alpha}$ and $b_{l,\alpha}$:

$$(P + \Phi)_{l,\alpha} = (2l + 1)^{-1}\left\{(r + \delta)^l\left[a_{l,\alpha} + \int_0^r \chi_{l,\alpha}(r')(r' + \delta)^{1-l}dr'\right]\right.$$
$$\left. - (r + \delta)^{-(l+1)}\left[b_{l,\alpha} + \int_0^r \chi_{l,\alpha}(r')(r' + \delta)^{l+2}dr'\right]\right\}\,. \tag{4.9}$$

The constants $a_{l,\alpha}$ and $b_{l,\alpha}$ are most economically determined by using the modal momentum equation (3.14) and requiring that $\partial\omega_{l,\alpha}/\partial t|_{r=0} = \partial\omega_{l,\alpha}/\partial t|_{r=1} = 0$. The constants $a_{l,\alpha}$ and $b_{l,\alpha}$ depend on the values of $\partial\omega_{l,\alpha}/\partial r$ and $\psi_{l,\alpha}$ at the two boundaries. One might think that it would be easier to solve numerically equation (3.18) by replacing ∇^2 with its finite-difference operator, a matrix, and then multiplying $\chi_{l,\alpha}h^{l,\alpha}$ by the inverse of that matrix. This technique avoids having to use the momentum equation to obtain the value of $(P + \Phi)_{l,\alpha}$ at the boundaries. However, this method is unusable; we cannot accurately construct the finite-difference operator for ∇^2 without first knowing the boundary conditions of its argument.

b) *The Sphere* ($\delta = 0$)

The boundary equations of a spherical shell in the limit $\delta \to 0$ are more restrictive than those of a sphere. At the center of the sphere our only boundary conditions are regularity and single-valuedness of the velocity, temperature, temperature gradient, pressure, and stress. In particular, we do not require that the velocity vanish at the origin.

If the modal equations (3.13)–(3.18) were linearized such that the turbulent pressure, viscosity, and conduction were set equal to zero (mean field-approximation) and if $\tilde{H} = 0$, then we can easily show that at the origin $\omega_{l,\alpha}$, $\psi_{l,\alpha}$, $T_{l,\alpha}$, $\Phi_{l,\alpha}$, and $P_{l,\alpha}$ each have leading order r^l and that $\partial\langle T\rangle/\partial r$ has leading order r (Chandrasekhar 1961). Furthermore, if $\langle H\rangle$ were an even function of r, we could also show that $(r^{-l}\omega_{l,\alpha})$, $(r^{-l}\psi_{l,\alpha})$, $(r^{-l}\Phi_{l,\alpha})$, $(r^{-l}P_{l,\alpha})$, $(r^{-l}T_{l,\alpha})$ and $(r^{-1}\partial\langle T\rangle/\partial r)$ are all even functions of r.

For example, we could write $\omega_{l,\alpha}$ in the form

$$\omega_{l\alpha} = r^l \sum_{i=0}^{\infty} S_i r^{2i}\,, \tag{4.10}$$

where the S_i are constants. Clearly for an arbitrary value of $\tilde{H}$ the r^l behavior of the modes at the origin does not apply. Even if $\tilde{H}$ were set to zero, it would not be obvious that the inclusion of the nonlinear terms in the modal equations would require r^l behavior at the origin. However, if for all l, the leading order of $H_{l,\alpha}$ is $\sim r^{l+\gamma}$ (with $\gamma \geq 0$) or if $H_{l,\alpha} = 0$, then we can rigorously prove that the nonlinear equations yield values of $\omega_{l,\alpha}$, $T_{l,\alpha}$, $\Phi_{l,\alpha}$, $\psi_{l,\alpha}$, and $P_{l,\alpha}$ with leading order $\sim r^l$ and $\partial\langle T\rangle/\partial r$ with leading order r. If we further restrict $(r^{-l}H_{l,\alpha})$ and $\langle H\rangle$ to be even functions of r, we can prove that $(r^{-l}\omega_{l,\alpha})$, $(r^{-l}\psi_{l,\alpha})$, $(r^{-l}T_{l,\alpha})$, $(r^{-l}P_{l,\alpha})$, $(r^{-l}\phi_{l,\alpha})$, and $(r^{-1}\partial\langle T\rangle/\partial r)$ are also even functions of r. In Paper II, we calculate solutions of the nonlinear modal equations with the latter, more restrictive boundary conditions.

Note that although the boundary conditions require that $\tilde{T}$, $\tilde{P}$, $\tilde{\phi}$, and the toroidal component of the velocity are zero at the origin, the $l = 1$ dipole part of the poloidal velocity is not necessarily zero. Knowing the leading-order terms of quantities as a function of l is important. If we did not determine the boundary conditions as a function of l, we would know only that $\omega \sim r^1$ at the origin. If we used the boundary condition $\omega_{l,\alpha} \sim r^1$ to construct difference operators for all of our modes, regardless of l, our numerical results would be incorrect. Similarly,

[1] The traditional Nusselt number, which is normally used to measure the efficiency of convection, is not well defined with our boundary conditions.

knowing that $(r^{-l}\omega_{l,\alpha})$ is an even function of r is important because second-order accurate finite-difference equations require knowledge of the first *and* the second leading-order terms at the origin. The boundary conditions show that a modal analysis can be very beneficial; methods of solving the Boussinesq equations in which we finite-difference in two or three spatial dimensions cannot use the fact that $(r^{-l}\omega_{l,\alpha})$ is an even function of r, and the solutions computed by these methods must reflect this loss of information.

We now outline the proof of the boundary conditions at $r = 0$ when $(r^{-l}H_{l,\alpha})$ and $\langle H\rangle$ are even functions of r. Consider the simple diffusion equation, $\partial\eta_{l,\alpha}/\partial t = \nabla_s^2\eta$, where $\eta_{l,\alpha}$ and all of its time derivatives are nonsingular at the origin. We must be able to write $\eta_{l,\alpha}$ in the form

$$\eta_{l,\alpha} = \sum_{i=0}^{\infty} S_i r^{(l+2i)}, \tag{4.11}$$

where the S_i are constants: if $\eta_{l,\alpha}$ contains a term proportional to r^β where $\beta \neq (l + 2i)$, then $\partial^{[(\beta+1)/2]ip}\eta_{l,\alpha}/\partial t^{[(\beta+1)/2]ip}$ is singular at the origin and violates our original assumptions. $[\ ]_{ip}$ is the operator which takes the greatest integer part of the quantity inside the brackets. If we wished to prove the boundary conditions, we would show that the nonlinear modal equations (3.13)–(3.18) that govern $\omega_{l,\alpha}$, $T_{l,\alpha}$, and $\psi_{l,\alpha}$ produce singularities in the high-order time derivatives unless the boundary conditions are satisfied.

It is interesting to see how the nonlinear terms in the modal equations preserve the boundary conditions. It is the selection rules for the B numbers and C numbers that make the inclusion of the nonlinear terms feasible. For example, equations (3.16)–(3.17) show that $\partial\psi_{l,\alpha}/\partial t$ is proportional to the nonlinear term, $r^{-1}\sum\psi_{l',\alpha'}\psi_{l'',\alpha''}\times B(h^{l,\alpha}, h^{l',\alpha'}, h^{l'',\alpha})$; it is therefore necessary that this nonlinear term has leading order $\sim r^l$ and that the product of this term with r^{-l} be an even function of r. Using the fact that $\psi_{l',\alpha'} \sim r^{l'}$, the leading order of the nonlinear term is $\sim r^\alpha$, where α is the smallest possible value of $(l' + l'' - 1)$. The index α can be determined from the triangle inequality for the l's that appear in $B(h^{l,\alpha}, h^{l'\alpha'}, h^{l''\alpha''})$ and is shown to be equal to l. It is the triangle inequality for the B number that ensures that the nonlinear term has the correct leading order. Similarly, it is the parity rule for B that makes the product of the nonlinear term with r^{-l} an even function of r.

V. MULTIPLICATION OF BASIS FUNCTIONS

a) $R^{l,m}$ *and* $I^{l,m}$ *Planforms*

To obtain solutions to the modal equations it is necessary that we find a suitable set of basis functions, determine their multiplicative properties, and compute their C numbers and B numbers. The spherical harmonics are the obvious choice of basis functions in which to expand the (θ, φ)-dependence of the fluctuating fields. However, for numerical computation it is advantageous to separate the real and the imaginary parts of the spherical harmonics by defining a new set of real basis functions, or planforms, $R^{l,m}$ and $I^{l,m}$:

$$R^{l,m}(\theta,\varphi) \equiv (2\pi)^{1/2}(Y^{l,m} + (-)^m Y^{l-m}) \qquad \text{for } l \geq m > 0, \tag{5.1}$$

$$I^{l,m}(\theta,\varphi) \equiv i(2\pi)^{1/2}(Y^{l,m} - (-)^m Y^{l-m}) \qquad \text{for } l \geq m > 0, \tag{5.2}$$

$$R^{l,0}(\theta,\varphi) \equiv (4\pi)^{1/2} Y^{l,0} \qquad \text{for } l > 0. \tag{5.3}$$

Notice that the $R^{l,m}$ are defined for $0 < l$ and $0 \leq m \leq l$ and the $I^{l,m}$ are defined for $0 < l$ and $0 \leq m \leq l$. (N.B. There is no $I^{l,m=0}$ planform.) Determining the multiplicative properties of the $R^{l,m}$ and $I^{l,m}$ planforms is equivalent to computing their B and C numbers. We find that

$$C(R^{l,m}, R^{l',m'}, I^{l'',m''}) = C(I^{l,m}, I^{l',m'}, I^{l'',m''}) = 0, \tag{5.4}$$

which means that the product of two R planforms is a sum of R planforms, the product of two I planforms is a sum of R planforms, and the product of an R and an I planform is a sum of I planforms. We also find that

$$B(R^{l,m}, R^{l',m'}, R^{l'',m''}) = B(R^{l,m}, I^{l',m'}, I^{l'',m''}) = 0. \tag{5.5}$$

Other relations among the B and C numbers are given in the Appendix.

The nonvanishing C numbers are computed in terms of the Wigner $3j$ symbols,

$$\begin{pmatrix} l & l' & l'' \\ m & m' & m'' \end{pmatrix}:$$

$$\begin{aligned} &C(R^{l',m'}, R^{l'',m''}, R^{l,m}) \\ &\quad = 2^{-1/2}n_1(m',m'')[(2l+1)(2l'+1)(2l''+1)]^{1/2}\begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \\ &\quad \times \left[\delta_{(m'+m''),m}(-1)^m\begin{pmatrix} l' & l'' & l \\ m' & m'' & -m \end{pmatrix} + \delta_{|m'-m''|,m}(-1)^{m_>}n_2(m',m'')\begin{pmatrix} l' & l'' & l \\ m' & -m'' & (m''-m') \end{pmatrix}\right], \end{aligned} \tag{5.6}$$

$$C(I^{l',m'}, I^{l'',m''}, R^{l,m})$$
$$= 2^{-1/2} n_1(m', m'')[(2l+1)(2l'+1)(2l''+1)]^{1/2} \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix}$$
$$\times \left[-\delta_{(m'+m''),m}(-1)^m \begin{pmatrix} l' & l'' & l \\ m' & m'' & -m \end{pmatrix} + \delta_{|m'-m''|,m}(-1)^{m_>} n_2(m'.\, m'') \begin{pmatrix} l' & l'' & l \\ m' & -m'' & (m''-m') \end{pmatrix} \right], \quad (5.7)$$

where $m_>$ is the greater of m' and m'' (if $m' = m$, then $m_> = m' = m''$) and

$$n_1(m', m'') = \begin{Bmatrix} 1 & m' \neq 0 \quad \text{and} \quad m'' \neq 0 \\ 2^{-1/2} & m' = 0 \quad \text{or} \quad m'' = 0 \end{Bmatrix}, \quad (5.8)$$

$$n_2(m', m'') = \begin{Bmatrix} 1 & m' \neq m'' \quad \text{or} \quad m' = m = 0 \\ (2)^{1/2} & m' = m'' \neq 0 \end{Bmatrix}. \quad (5.9)$$

The fact that

$$\begin{pmatrix} l & l' & l'' \\ m & m' & m'' \end{pmatrix}$$

is zero unless $(m + m' + m'') = 0$ implies that $C(R^{l,m}, R^{l',m'}, R^{l'',m''})$ and $C(R^{l,m}, I^{l',m'}, I^{l'',m''})$ are nonzero only if $m = |m' \pm m''|$. Furthermore, for C to be nonzero, l, l', and l'' must obey the C-number parity rule and triangle inequality. The nonvanishing B numbers, $B(R^{l,m}, R^{l',m'}, I^{l'',m''})$ and $B(I^{l,m}, I^{l',m'}, R^{l'',m''})$, can be computed in terms of the C numbers. These formulae are found in the Appendix. For a B number to be nonzero, no two of its arguments may be identical, the l's must obey the B-number parity rule and triangle inequality, m must be equal to $|m' \pm m''|$, and m, m', and m'' cannot all be equal to zero.

b) *Symmetries and Closed Sets*

An exact solution to the modal Boussinesq equations is not necessarily comprised of a complete set of basis functions. It is necessary, however, that the set of basis functions that appears in an exact solution be closed with respect to the multiplications that occur in the nonlinear terms of the modal equations. Let S be an orthonormal set of basis functions with a subset S'. S' is defined to be *closed* if and only if for all $h^{l',\alpha'}, h^{l'',\alpha''} \in S'$, the products $(h^{l',\alpha'} \cdot h^{l'',\alpha''})$ and $[(\nabla_s h^{l',\alpha'} \times \nabla_s h^{l'',\alpha''}) \cdot \hat{e}_r]$ are spanned by the elements of S'. Equivalently, S' is closed if and only if $C(h^{l,\alpha}, h^{l',\alpha'}, h^{l'',\alpha''}) = B(h^{l,\alpha}, h^{l',\alpha'}, h^{l'',\alpha''}) = 0$ for all $h^{l,\alpha}, h^{l',\alpha'}, h^{l'',\alpha''} \in S$ with $h^{l',\alpha'}, h^{l'',\alpha''} \in S'$ and $h^{l,\alpha} \notin S'$. An exact solution is composed of a closed set of basis functions. (For the exception to this rule see § VII.) Each closed, proper subset corresponds to a physical symmetry that is shared by the basis functions that belong to the subset and that is preserved under multiplication. Symmetric convection with patterns in the form of rolls, hexagons, and rectangles are known to exist in the plane-parallel geometry of laboratory experiments with a wide range of Rayleigh numbers (Whitehead and Parsons 1978). There is analytic evidence (Busse 1975) that at the onset of convection in spherical shells, symmetric solutions are preferred. The Boussinesq equations with spherical boundary conditions can have a solution with a velocity field that (1) is symmetric with respect to inversion through the origin, (2) is invariant with respect to reflection about one (or more orthogonal) plane(s) that pass through the origin, (3) is invariant about rotations about one axis, or (4) has the same rotational symmetries as one of the five regular solids. In Paper II we examine the relative stability of these highly symmetric solutions. To understand plane-parallel convection in rolls or with hexagonal cells, it is useful to know how to construct roll-like or hexagonal planforms out of sines and cosines. Similarly, to study spherical convection that has a plane of reflection symmetry or with dodecahedral cells, we need to know how to construct reflection-symmetric and dodecahedral planforms in terms of the spherical harmonics.

c) *Inversion, Reflection, and Simple Rotational Symmetries*

Solutions that are inversion symmetric have scalar quantities, $\tilde{\omega}$, $\tilde{P}$, $\tilde{\phi}$, $\tilde{T}$, and $\tilde{\rho}$, that consist only of planforms, $R^{l,m}$ and $I^{l,m}$, with even l's and pseudo-scalars $\tilde{\psi}$ that consist only of planforms with odd l's. A solution that is invariant under reflection about the $y = 0$ plane has scalars composed only of R basis functions and has pseudo-scalars composed only of I basis functions. A solution that is reflection-symmetric with respect to the $x = 0$ plane has scalars composed of $R^{l,m=\text{even}}$ and $I^{l,m=\text{odd}}$ basis functions and pseudo-scalars composed of $R^{l,m=\text{odd}}$ and $I^{l,m=\text{even}}$ basis functions. A solution with reflection symmetry about the $z = 0$ plane requires $R^{l,m}$ and $I^{l,m}$ basis functions with $(l + m) =$ even for the scalars and $(l + m) =$ odd for the pseudo-scalars. A solution invariant under rotations about the z-axis by $2\pi/\hat{m}$ (with $\hat{m}$ integral) has scalars and pseudo-scalars made from the set of basis functions, $R^{l,m}$, $I^{l,m}$ with m equal to an integral multiple of $\hat{m}$. The selection rules for the C and B numbers and equations (5.4)–(5.5) are sufficient to show that a fluid which initially has a reflection, rotation, or inversion symmetry always maintains that symmetry and that the nonlinear terms in the Boussinesq equations never give rise to symmetry-breaking terms. If a solution has reflection symmetry about a plane that is not orthogonal to a principal axis, the scalars and pseudo-scalars will be made up of all of the $R^{l,m}$ and $I^{l,m}$ basis functions and the symmetry will be "hidden." By rotating the solution into a "preferred frame" or equivalently by choosing a better tailored set of basis functions, we can reduce the number of modal equations needed in the solution.

d) *Polyhedral Harmonics*

There are closed sets of planforms that have the rotation symmetries of the five regular solids: the tetrahedron, hexahedron, octahedron, dodecahedron, and icosahedron.

A velocity field that is invariant under the rotations that leave a dodecahedron invariant has scalars and pseudo-scalars made from planforms that belong to the closed set of dodecahedral harmonics. This set of harmonics can be obtained by finding the set of orthonormal functions that are simultaneously eigenfunctions of ∇_s^2 and the dodecahedral rotation operators. (This process is explicitly demonstrated by Marcus 1978, who also has a complete discussion of the rotation operators that leave the regular solids invariant.) The set of dodecahedral harmonics is unique up to rotations; however, as soon as the orientation of any one of the harmonics is chosen, the orientation (but not sign) of all of the other harmonics in the closed set becomes fixed. Although each dodecahedral harmonic has a unique value of l, there are not dodecahedral harmonics for all values of l. For example, there are not dodecahedral harmonics with l less than 6. The fundamental dodecahedral harmonic has with $l = 6$ and can be written in one orientation as

$$d^{l=6}(\theta, \varphi) = (11)^{1/2}/5[R^{6,0} - (14/11)^{1/2}R^{6,5}]$$
$$= 2(11\pi)^{1/2}/5[Y^{6,0} - (7/11)^{1/2}(Y^{6,5} - Y^{6,-5})] . \tag{5.10}$$

Using the fact that dodecahedrons are invariant under inversion followed by a reflection about one of its mid-planes, we can show that all dodecahedral harmonics, with the orientation implied by equation (5.10), are linear combinations of $R^{l=\text{even},m}$ or $I^{l=\text{odd},m}$ planforms where m is a multiple of 5. Closure, of course, guarantees that the product of any two dodecahedral harmonics is a linear sum of dodecahedral harmonics.

For the calculation in Paper II, we have computed the first 200 dodecahedral, tetrahedral, and hexahedral harmonics along with their B and C numbers. The orientation of our computed polyhedral harmonics is arbitrary, but it is only their B and C numbers that ever appear in the modal equations, and the B and C numbers are rotationally invariant quantities and are therefore independent of our arbitrary orientation. Since icosahedrons are dual to dodecahedrons and they are invariant under the same rotation operators, the isocosahedral harmonics are identical to the dodecahedral harmonics. Similarly, duality makes the octahedral harmonics identical to the hexahedral harmonics. Although the tetrahedron is dual only to itself, it can be shown that the hexahedral harmonics are a closed, proper subset of the tetrahedral harmonics. (Octahedrons and hexahedrons are invariant under all rotations that leave the tetrahedron invariant, but they have a plane of reflection symmetry that tetrahedrons do not have.) The fundamental tetrahedral harmonic has $l = 3$ and can be written

$$t^3 = \tfrac{1}{3}(5)^{1/2}[R^{3,0} - 2(\tfrac{1}{5})^{1/2}R^{3,3}] = \tfrac{2}{3}(5\pi)^{1/2}[Y^{3,0} - (\tfrac{2}{5})^{1/2}(Y^{3,3} - Y^{3,-3})] . \tag{5.11}$$

The $l = 4$ tetrahedral harmonic can be written

$$t^4 = -\tfrac{1}{3}(7/3)^{1/2}[R^{4,0} - 2(\tfrac{5}{7})^{1/2}R^{4,3}] = -\tfrac{2}{3}(7\pi/3)^{1/2}[Y^{4,0} + (10/7)^{1/2}(Y^{4,3} + Y^{4,-3})] , \tag{5.14}$$

and it can be shown that t^4 is the hexahedral harmonic with the lowest value of l.

It is tempting to compare the Christopherson (1940) hexagonal planforms h^{hex}, that are observed in plane-parallel geometries with the polyhedral harmonics of spherical geometries. The outstanding feature of h^{hex} is that its square is equal to a linear sum of its shorter wavelength overtones *plus itself*, i.e., $C(h^{\text{hex}}, h^{\text{hex}}, h^{\text{hex}}) \neq 0$. Some of the polyhedral harmonics share this feature, but none of the polyhedral harmonics with odd values of l can have this property due to the l-parity rule for C numbers. Because the closed sets of hexagonal planforms, dodecahedral harmonics, hexahedral harmonics, and tetrahedral harmonics have qualitatively different sets of C and B numbers, the convective solutions with these respective symmetries are also qualitatively different. There are other differences between hexagonal and polyhedral planforms. In a plane-parallel geometry with infinite horizontal domain, the hexagonal planforms have a continuous spectrum of allowable horizontal wavelengths that range in size from 0 to ∞; in spherical geometries the polyhedral harmonics have a discretized spectrum of wavelengths that has a finite upper bound. The discreteness and finite upper bound of the allowable horizontal wavelengths have many consequences. For example: if a linear stability analysis shows that the horizontal wavelength of the fundamental convective mode in a spherical geometry is greater than $2\pi(42)^{-1/2}$, then we know immediately that the convective solution cannot have dodecahedral symmetry. In a plane-parallel geometry, there is no restriction of the horizontal wavelength of the fundamental convective mode that can ever prohibit an equilibrium solution from having hexagonal symmetry.

VI. ANGULAR MOMENTUM

a) *Transfer of Angular Momentum among Radial Shells*

The angular momentum in a thin shell of radius r and thickness dr (to first order in $\Delta\rho/\rho_0$) is:

$$j(r) = 4\pi(r + \delta)^3\langle \hat{e}_r \times v\rangle dr . \tag{6.1}$$

The quantity $\boldsymbol{J} \equiv \int_0^1 \boldsymbol{j}(r)$ is exactly conserved by the Boussinesq equations, and to first order in $\Delta\rho/\rho_0$ it is equal to the total angular momentum of the fluid. Equation (6.1) shows that the poloidal part of the velocity does not contribute to $\boldsymbol{j}(r)$ and that only the $l = 1$ component of the toroidal velocity affects $j(r)$. Therefore, $\boldsymbol{j}(r)$ and $\boldsymbol{J}$ are purely poloidal fields. In terms of the $l = 1$ component of the pseudo-scalar generator, $\psi_{l=1}$, the angular momentum is

$$\boldsymbol{j}(r) = 4\pi(r + \delta)^3\langle \boldsymbol{\nabla}_s\psi_{l+1}\rangle dr\,. \tag{6.2}$$

With the notation

$$\psi_{l=1} \equiv \psi_{1R0}R^{1,0} + \psi_{1R1}R^{1,1} + \psi_{1I1}I^{1,1}\,, \tag{6.3}$$

we obtain

$$\boldsymbol{j}(r) = 8\pi(3)^{-1/2}(r + \delta)^3dr(\psi_{1R0}\hat{e}_z - \psi_{1R1}\hat{e}_x + \psi_{1I1}\hat{e}_z)\,. \tag{6.4}$$

It should be noted that none of the polyhedral harmonics have an $l = 1$ component and therefore solutions with polyhedral symmetries never have angular momentum in any of their shells.

To transport angular momentum among different shells, it is necessary that $\partial\psi_{l=1}/\partial t \neq 0$. From equation (3.16) we see that the $\psi_{l=1}$ mode is affected only by the viscous dissipation of itself, Pr $\nabla^2\psi_{l=1}$, and by nonlinear eddy-eddy interactions. The nonlinear interaction between the two toroidal eddies generated from $\psi_{l',\alpha'}h^{l',\alpha'}$ and $\psi_{l'',\alpha''}h^{l'',\alpha''}$ affects $\partial\psi_{l=1,\alpha}/\partial t$ only if $B(h^{1,\alpha}, h^{l',\alpha'}, h^{l'',\alpha''}) \neq 0$. For the B number to be nonzero, l' must equal l''. We showed in § III*c* that if $l' = l''$, the toroidal-toroidal contribution to $\partial\psi/\partial t$ vanishes. Therefore, the toroidal-toroidal nonlinear interaction never contributes to $\partial\boldsymbol{j}(r)/\partial t$ or changes the angular momentum of a radial shell. The nonlinear interaction of the poloidal eddies generated by $\omega_{l',\alpha'}h^{l',\alpha}$ and $\omega_{l'',\alpha''}h^{l'',\alpha''}$ affects $\partial\boldsymbol{j}(r)/\partial t$ only if $l' = l''$ and only if $\omega_{l',\alpha'}$ is not proportional to $\omega_{l'',\alpha''}$. Therefore, the self-interaction of a velocity field generated from a single mode or planform never changes the angular momentum of a shell. The interaction between the poloidal field generated from $\omega_{l',\alpha'}h^{l',\alpha'}$ and the toroidal field generated from $\psi_{l'',\alpha''}h^{l'',\alpha''}$ changes $\boldsymbol{j}(r)$ only if $l' = l'' \pm 1$. The viscous term, Pr $\nabla^2\psi_{l=1}$, can transfer angular momentum among spherical shells; but if the C and B numbers of the eddies are near unity and if their Reynolds numbers are large, then the viscous transport of angular momentum is overwhelmed by the transport due to the nonlinear interaction or turbulent viscosity. In stellar interiors the mechanism most responsible for radial transport of angular momentum and the subsequent differential rotation is the turbulent viscosity. If a star were initially differentially rotating such that the velocity was purely toroidal, then all of the angular momentum would be transported by viscosity alone since the toroidal-toroidal interaction cannot change $\boldsymbol{j}(r)$. As soon as poloidal advection began in the star, the radial transport of angular momentum would be greatly enhanced by the turbulent viscosity.

b) *Conservation of Total Angular Momentum in a Truncated Set of Modal Equations*

With proper boundary conditions, not only does the infinite set of coupled Boussinesq modal equations conserve total angular momentum but also any arbitrarily truncated set of modal equations exactly conserves $\boldsymbol{J}$. We need to examine the conservation of $\boldsymbol{J}$ explicitly so that we can be certain that our numerical treatment of the boundaries and difference operators leads to an exact conservation of $\boldsymbol{J}$. The derivative $\partial\boldsymbol{J}/\partial t$ is proportional to

$$\int_0^1 \partial\psi_{l=1,\alpha}/\partial t(r + \delta)^3dr$$

which has contributions from (1) the toroidal-poloidal interaction of the fields generated from $\psi_{l',\alpha'}h^{l',\alpha'}$ and $\omega_{l'',\alpha''}h^{l'',\alpha''}$ for all $h^{l',\alpha'}$ and $h^{l'',\alpha''}$ that belong to the truncated set of modes; (2) the poloidal-poloidal interaction of the fields generated from $\omega_{l',\alpha'}h^{l',\alpha'}$ and $\omega_{l'',\alpha''}h^{l'',\alpha''}$ for all $h^{l',\alpha'}$ and $h^{l'',\alpha''}$ that belong to the set of truncated modes; and (3) the viscous dissipation, Pr $\nabla^2\psi_{l=1,\alpha}$. To prove that the truncated modal equations conserve $\boldsymbol{J}$, it is sufficient to show that each of these three contributions to

$$\int_0^1 \partial\psi_{l=1,\alpha}/\partial t(r + \delta)^3dr$$

separately vanish. The poloidal-toroidal contribution from $h^{l',\alpha'}$ and $h^{l'',\alpha''}$ is:

$$\int dr[\![[l'(l' + 1) - l''(l'' + 1) - 2]l'(l' + 1)\{\omega_{l',\alpha'}(r + \delta)\partial[(r + \delta)\psi_{l'',\alpha''}]/\partial r\}$$
$$- [l''(l'' + 1) - l'(l'' + 1) - 2]l''(l'' + 1)\{\psi_{l'',\alpha''}(r + \delta)\partial[(r + \delta)\omega_{l',\alpha'}]/\partial r]\!]C(h^{l,\alpha}, h^{l',\alpha'}, h^{l'',\alpha''})\,. \tag{6.5}$$

Setting l' equal to $l'' \pm 1$ (for C number to be nonzero) and integrating (6.5) by parts, we find that only the surface terms survive. The surface terms vanish if $\omega = 0$ at $r = 0$ and $r = 1$. Therefore, in a sphere or spherical shell with impermeable boundaries the poloidal-toroidal interaction of the $h^{l',\alpha'}$ and $h^{l'',\alpha''}$ modes do not contribute to $\partial\boldsymbol{J}/\partial t$.

To conserve angular momentum exactly in a numerical computation it is necessary that our difference-operators are chosen such that an integration by parts leaves *no* round-off errors. By a similar integration by parts we find that the poloidal-poloidal interaction does not change $\boldsymbol{J}$ if $\omega = 0$ at the boundaries. The viscous contribution to

$$\int_0^1 \partial\psi_{l=1,\alpha}/\partial t(r+\delta)^3 dr\,,$$

after integration by parts is

$$\Pr\,[_0{}^1(r+\delta)^4\partial[\psi_{l=1,\alpha}/(r+\delta)]/\partial r\,. \tag{6.6}$$

The zero-stress boundary condition requires that $\partial[\psi_{l=1,\alpha}/(r+\delta)]/\partial r = 0$ at $r = 0$ and $r = 1$. (If $\delta = 0$, regularity requires that $\partial[\psi_{l=1,\alpha}/(r+\delta)]/\partial r = 0$ at the origin.) Therefore, the no-stress boundary condition is needed to conserve angular momentum. A no-slip boundary allows $\partial[\psi_{l=1,\alpha}/(r+\delta)]/\partial r \neq 0$ at the boundaries, the boundary exerts torques on the fluid, and angular momentum is not conserved. It is necessary that our numerical treatment of the boundaries sets $\partial[(\psi_{l=1,\alpha}/(r+\delta)]/\partial r$ equal to zero with *no* round-off errors at $r = 1$ and $r = 0$ in order that $\boldsymbol{J}$ be exactly conserved.

VII. AN EXAMPLE: SOLID BODY ROTATION

At this point, it is instructive to work out an analytic but trivial solution to the modal equations. The example demonstrates the nonlinear interactions and also shows how to choose the constants of integration that appear in $P_{l,\alpha}$ when we compute $\tilde{P}$ from Poisson's equation (4.9). Solid-body rotation (about the z-axis) in the language of modes is: $\omega_{l,\alpha} = \tilde{\Phi}_{l,\alpha} = \tilde{T} = 0$ for all l, α; $\partial\langle T\rangle/\partial r = -\mathscr{L}/(r+\delta)^3$; and $\psi = c(r+\delta)R^{1,0}$, where c is a constant. Does this solution satisfy the modal equations? The equations for $\partial\langle T\rangle/\partial r$, $\tilde{\Phi}$, and $\tilde{T}$ are easily satisfied with $\partial\tilde{T}_{l,\alpha}/\partial t = 0$ for all l, α. Because there is no viscous dissipation in solid-body rotation and because there is no contribution to $\partial\psi/\partial t$ from a single, self-interacting toroidal mode (see § III*c*), the modal equation for $\partial\psi_{l,\alpha}/\partial t$ is satisfied with $\partial\psi_{l,\alpha}/\partial t = 0$ for all l, α. The selection rules for C numbers show that for all components of ω, other than the $R^{2,0}$ component, $\partial\omega_{l,\alpha}/\partial t = 0$. For the $R^{2,0}$ component,

$$\partial\omega_{2R0}/\partial t = -\tfrac{1}{6}(r+\delta)\partial P_{2R0}/\partial r - (5)^{-1/2}c^2(r+\delta)^2/3\,. \tag{7.1}$$

The term in equation (7.1) that is proportional to c^2 is due to the toroidal-toroidal interaction. Plugging the solid-body rotation values of ω, ψ, Φ, and T into the modal equation for $P_{l,\alpha}$, we find that all of the components of the turbulent pressure, except for the $R^{2,0}$ component, vanish due to the C number selection rules. The $R^{2,0}$ component of the turbulent pressure, $\langle R^{2,0}\nabla\cdot[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}]\rangle$, vanishes because $\psi_{l=1}$ is linear in the radial coordinate. Therefore,

$$\nabla^2(P_{l,\alpha}h^{l,\alpha}) = 0 \qquad \text{for all } l, \alpha\,. \tag{7.2}$$

If the fluid were differentially rotating, i.e., $\psi = c(r)(r+\delta)R^{2,0}$ with $c(r)$ not equal to a constant, then $\nabla^2(P_{2R0}R^{2,0})$ would not be equal to zero. Equation (7.2) requires that $\tilde{P}$ be equal to its homogeneous solution,

$$\tilde{P} = \sum_{l,\alpha}[a_{l,\alpha}(r+\delta) + b_{l,\alpha}(r+\delta)^{-l-1}]h^{l,\alpha}\,.$$

The constants of integration, $a_{l,\alpha}$ and $b_{l,\alpha}$, are chosen so that

$$\partial\omega/\partial t|_{r=0} = \partial\omega/\partial t|_{r=1} = 0\,. \tag{7.3}$$

From equation (7.1) we see that

$$\tilde{P} = -c^2(5)^{-1/2}(r+\delta)^2R^{2,0}/6\,. \tag{7.4}$$

With this pressure, $\partial\omega/\partial t$ equals zero not only at the boundaries but for all values of r. Therefore, solid-body rotation is a steady-state equilibrium solution to the modal Boussinesq equations. If the fluid were differentially rotating, ψ would not be linear in r, $\nabla^2\tilde{P}$ would not be equal to zero, the pressure term in equation (7.1) would not balance the toroidal-toroidal interaction, and $\partial\omega_{2R0}/\partial t$ would not equal zero. Therefore, differential rotation causes the growth of an $R^{2,0}$ component of the poloidal velocity and the growth of meridional circulation. (This circulation will be discussed in a later paper.) Before leaving this example we note that the fluctuating pressure is proportional to $R^{2,0}$ or $Y^{2,0}$ and that the total pressure, $\langle P\rangle + \tilde{P}$, at a given radius is greater at the equator than it is at the poles. If we had chosen to make the outer boundary of the fluid the locus of points where $P = 0$, then the $Y^{2,0}$ nature of $\tilde{P}$ would make an outer boundary of the fluid an oblate spheroid—a well-known fact for a rotating incompressible fluid.

VIII. TRUNCATION OF THE MODAL EQUATIONS

Except in rare cases, such as solid-body rotation, the nonlinear terms in the Boussinesq equations require that a solution be represented by an infinite, closed set of basis functions. To compute numerical solutions to the modal equations it is necessary to arbitrarily truncate the modal equations (Galerkin's method) and produce only approximate equilibrium solutions. With this type of approximation, it is generally not possible to provide a rigorous, analytic error bound for P, T, ρ, Φ, and $\boldsymbol{v}$. However, by appealing to the physical interpretation of the modal equations and by use of numerical experimentation, we can provide some justification of our truncations. Since selecting the finite set of basis functions determines what horizontal wavelengths we are allowing in the solution, we review the relevant length-scales that might appear in a fluid according to two possible pictures of convection. By definition of a Boussinesq fluid, the pressure scale-height is larger than any physical length that appears in the fluid and we may ignore it. The largest physical length is $(r_2 - r_1)$, which is probably the length scale of the largest convective eddies. The largest eddies are the ones that are most strongly driven by the convective buoyancy. At smaller wavelengths, the buoyancy forces become dominated by inertial forces; the kinetic energy of the flow no longer increases at the expense of the energy liberated by the buoyant bubbles; and temperature becomes a passive scalar. Another small scale is the critical wavelength from the shear instability caused by the strains of the large convective eddies; below this wavelength we might expect the fluid to be nearly isotropic. Furthermore, there is a cutoff length scale such that eddies smaller than this cutoff have no major role in transporting energy. For eddies smaller than all of these length scales, it is possible that the fluid has an inertial-convective subrange and that there is a smooth energy cascade (Tennekees and Lumley 1972).

According to classical theory, at still smaller wavelengths the Peclet numbers (Reynolds number times Prandtl number) of the eddies becomes less than unity and the flow develops a more nearly isothermal inertial-diffusive subrange. The smallest length in the problem is the Kolmogorov microscale where the energy cascade stops and the kinetic energy is efficiently dissipated. Superposed on this set of lengths are the one or more lengths that are characteristic of any boundary layers that form in the fluid.

The brute-force method of selecting the set of basis functions used in the modal equations is to choose a set that is complete for all wavelengths greater than a critical length, $2\pi[l_{\text{crit}}(l_{\text{crit}} + 1)]^{-1/2}$, such that this critical length is much less than all of the physical length scales of the convecting fluid. This set of basis functions is all $R^{l,m}$ and $I^{l,m}$ with $l < l_{\text{crit}}$. Presumably, any eddies excluded from a solution with this truncation decay rapidly due to viscosity and are unimportant in heat, momentum, and kinetic energy transport. Certainly this truncation is no worse than a three-dimensional finite-differencing scheme with a grid size of $2\pi[l_{\text{crit}}(l_{\text{crit}} + 1)]^{-1/2}$. One problem common to both the finite-differencing and Galerkin techniques is that although the equilibrium solutions may be well described for lengths greater than $2\pi[l_{\text{crit}}(l_{\text{crit}} + 1)]^{-1/2}$, we cannot determine if there are perturbations with wavelengths smaller than this resolution that cause an equilibrium solution to be unstable.

The brute-force method of choosing basis functions is feasible if the Reynolds numbers are not too large. In an extremely turbulent fluid, such as the convective interior of a star, a more subtle truncation is needed. In Papers III and IV we employ a technique where we use the modal equations to determine the solution for eddies with $l < l_{\text{is}}$, where $2\pi[l_{\text{is}}(l_{\text{is}} + 1)]^{-1/2}$ is a length that lies in the inertial subrange of the convective fluid. We use a spectral theory of turbulence to compute solutions for the eddies with $l \geq l_{\text{is}}$. The trick of the method is in matching the two solutions and showing that the total solution is independent of the exact value of l_{is}. The eddies that we examine with the modal part of the analysis are those that carry most of the heat flux, generate the kinetic energy, possess the interesting symmetries, have the dynamical interactions that determine the relative stability of one equilibrium solution over another, and (for the non-Boussinesq fluids studied in Papers III and IV) generate the convective overshoot.

It is possible that a classically turbulent fluid with a smooth energy spectrum is not an accurate description of high-Reynolds-number convection. There is some experimental evidence (Gollub and Swinney 1975; Whitehead and Parsons 1973) and some theoretical work (Ruelle and Takens 1971) that show that high-Reynolds-number convection is dominated by modal structure and that the energy spectrum is not a smooth function of wavelength. Papers II and IV report on self-consistent numerical experiments that attempt to differentiate between these two pictures of high-Reynolds-number convection, and thereby establish the best set of basis functions to include in the truncation.

IX. DISCUSSION

In this paper we have derived modal Boussinesq equations and their boundary conditions in spherical geometries. Two spatial dimensions are represented by modes leaving us with a set of partial differential equations with radius and time as the independent variables. All of this formalism would be somewhat pedantic if we limited our numerical calculations to a single mode or to only a few modes—the computations in Paper II include up to 200 modes. How does our modal analysis differ from previous attempts to solve the hydrodynamic equations that govern convection? The analyses of other authors who have used modes to represent two of the spatial dimensions have been done in plane geometries—either rectangular (see Gough, Spiegel, and Toomre 1975) or cylindrical (see Jones, Moore, and Weiss 1976). As we have stated, the crucial decision that must be made when using modal equations is which modes are needed to describe the relevant physics and which modes can be omitted. When using a plane geometry to approximate a sphere or spherical shell, it becomes more difficult to make this decision. If the plane horizontal boundaries are removed to infinity, the eigenmodes form a continuous rather than a discrete spectrum. How do we select a finite set of modes from a continuous spectrum? There is no such thing as the "200 largest wavelength modes" as there is in a spherical geometry. It is necessary to arbitrarily choose a set of modes with our favorite sizes and shapes. Our choice arbitrarily and permanently determines the relative spatial phases of the interacting modes—this loss of phase information can destroy the calculation. The imposition of finite horizontal boundaries of arbitrary shape, size, and type (i.e., no-slip or stress-free) also predetermine the relative spatial phases of the modes.

We must be somewhat circumspect in approximating a thin shell with a plane-parallel layer or approximating a polyhedral harmonic with a hexagonal planform. For example, the C number made of three identical Christopherson hexagonal modes, $C(h^{\text{hex}}, h^{\text{hex}}, h^{\text{hex}})$, is nonzero, whereas the C number made of the three fundamental tetrahedral modes, $C(t^3, t^3, t^3)$, is zero. Velocity fields with tetrahedral symmetry are different from velocity fields with hexagonal symmetry.

The physics of convection may qualitatively depend on the boundary conditions. Just as three-dimensional turbulence cannot be described by two-dimensional flows, there may be properties of convection, such as angular momentum transfer in a sphere, which may be indescribable in a plane-parallel geometry. (This last point will be discussed in a later paper.) Finally, our analysis has included the toroidal modes which are omitted in most calculations that are concerned with slowly rotating, nearly linear convective flows. Baker and Spiegel (1975) have included toroidal modes in their analysis of a rotating fluid using Cartesian coordinates. However, much of the physics of angular momentum transfer as well as the rotational symmetries are buried in the Cartesian notation and cannot be exploited.

Excellent work on convection in spherical shells has been done by Gilman and co-workers (Gilman 1975, 1977). Gilman performs modal calculations in which one spatial dimension (the longitude) is represented with modes. He uses a coupled set of partial differential equations that have *two* independent spatial coordinates as dependent variables. Gilman does not consider the case of a convecting sphere of fluid, but limits his discussion to shells. We consider shells, as well as spheres of fluid where the velocity at the origin may be nonzero.

Since we represent both the θ and the φ dependence with modes, our modes are eigenfunctions of both the total angular momentum operator, $r^{-2}\nabla_s^2$, and the z-axis angular momentum operator, $\partial^2/\partial z^2$; Gilman's planforms are eigenfunctions only of $\partial^2/\partial z^2$. Since the radial dependence of a mode at the origin is a function of its total angular momentum eigenvalue, we impose different boundary conditions for modes with different total angular momentum eigenvalues. If we used Gilman's method, we could not determine the total angular momentum eigenvalue of a mode (in fact, most modes would not be eigenfunctions of the total angular momentum operator) and we would have to use a much weaker boundary condition that would be general enough to apply to all of the modes. Like Baker and Spiegel's work in Cartesian coordinates, Gilman's modes partially hide the mechanics of angular momentum transfer and the rotational symmetries.

Gilman solves his equations with two-dimensional finite-differencing. The advantage of finite-differencing over modal analysis is that the former requires much less numerical work to compute nonlinear terms. However, it may require many more grid points to describe a flow that it requires modes. For example, if all of the physical properties of a convecting fluid can be described by the fundamental dodecahedral mode, then we need only one modal equation. How many hundreds of grid points would we need to describe this flow? Whatever advantage there may be in using modal analysis over finite-differencing will depend upon how much structure and symmetry there is in our flows and how well we can exploit that structure and symmetry.

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APPENDIX

To facilitate writing $B(R^{l,m}, R^{l',m'}, I^{l'',m''})$ and $B(I^{l,m}, I^{l',m'}, I^{l'',m''})$ in terms of the C-numbers, we define $S^{\pm}(l, m, l', m', l'', m'', p)$:

$$
\begin{aligned}
S^{\pm}&(l, m, l', m', l'', m'', p) \\
&\equiv \begin{pmatrix} l' & 1 & l' \\ m'+1 & -1 & -m' \end{pmatrix} \left\{ \begin{pmatrix} l\pm 1 & 1 & l \\ m-1 & 1 & -m \end{pmatrix} [n_5(m) C(R^{l',m'+1}, I^{l\pm 1,|m-1|}, I^{l'',m''}) \right. \\
&\qquad + n_6(m) C(R^{l\pm 1,|m-1|}, I^{l',m'+1}, I^{l'',m''})]\delta_{m'+m,m''} \\
&\qquad \left. + p \begin{pmatrix} l\pm 1 & 1 & l \\ m+1 & -1 & -m \end{pmatrix} n_5(m''+1) C(R^{l',m'+1}, I^{l+1,m+1}, I^{l'',m''})\delta_{|m'-m|,m''} \right\} \\
&+ \begin{pmatrix} l' & 1 & l' \\ m'-1 & 1 & -m' \end{pmatrix} \left\{ \begin{pmatrix} l\pm 1 & 1 & l \\ m+1 & -1 & -m \end{pmatrix} [n_5(m') C(R^{l\pm 1,m+1}, I^{l',|m'-1|}, I^{l'',m''}) \right. \\
&\qquad + n_6(m') C(R^{l',|m'-1|}, I^{l\pm 1,m+1}, I^{l'',m''})]\delta_{m'+m,m''} \\
&\qquad + p \begin{pmatrix} l\pm 1 & 1 & l \\ m-1 & 1 & -m \end{pmatrix} [-n_3(m, m') C(R^{l\pm 1,|m-1|}, I^{l',|m'-1|}, I^{l'',m''}) \\
&\qquad \left. + n_4(m, m') C(R^{l',]m'-1]}, I^{l\pm 1,|m-1|}, I^{l'',m''})]\delta_{|m'-m|,m''} \right\},
\end{aligned} \tag{A1}
$$

where

$$n_5(m) = \begin{Bmatrix} 2 & m \neq 1 \\ 0 & m = 1 \end{Bmatrix}, \tag{A2}$$

$$n_6(m) = \begin{Bmatrix} 0 & m \neq 1 \\ 2^{1/2} & m = 1 \end{Bmatrix}, \tag{A3}$$

$$n_4(m, m') = \begin{Bmatrix} 2^{1/2} & m' = 1, m \neq 1 \\ 0 & m' \neq 1 \text{ or } m = 1 \end{Bmatrix}, \tag{A4}$$

$$n_3(m, m') = \begin{Bmatrix} 0 & m = m', \text{ or } m' = 1 \\ 2^{1/2} & m = 1, m' \neq 1 \\ -2 & m = 0, m' > 1 \\ 2 & m > 1, m' \neq 1, m' \neq m \end{Bmatrix}. \tag{A5}$$

In terms of $S^{\pm}$, the B numbers are:

$$\begin{aligned} B(I^{l,m}, I^{l',m'}, I^{l'',m''}) = &-(-)^{m''+l''}\tfrac{1}{2}[l'l(l'+1)(l+1)]^{1/2}(2l'+1)^{1/2} \\ &\times \Big\{ l^{1/2}S^{+}(l, m, l', m', l'', m'', p = -1) + (l+1)^{1/2}S^{-}(l, m, l', m', l'', m'', p = -1) \\ &- 2n_7(m')\begin{pmatrix} l' & 1 & l' \\ m' & 0 & -m' \end{pmatrix}\Big[l^{1/2}\begin{pmatrix} l+1 & 1 & l \\ m & 0 & -m \end{pmatrix} C(I^{l+1,m}, R^{l',m'}, I^{l'',m''}) \\ &+ (l+1)^{1/2}\begin{pmatrix} l-1 & 1 & l \\ m & 0 & -m \end{pmatrix} C(I^{l-1,m}, R^{l',m'}, I^{l'',m''})\Big]\Big\}; \end{aligned} \tag{A6}$$

$$\begin{aligned} R(R^{l,m}, R^{l',m'}, I^{l'',m''}) = &(-)^{m''+l''}\tfrac{1}{2}[l'l(l'+1)(l+1)]^{1/2}(2l'+1)^{1/2}[n_7(m)n_7(m')]^{-1} \\ &\times \Big\{ l^{1/2}S^{+}(l, m, l', m', l'', m'', p = 1) + (l+1)^{1/2}S^{-}(l, m, l', m', l'', m'', p = 1) \\ &- n_7(m)n_5(m'+1)\begin{pmatrix} l' & 1 & l' \\ m' & 0 & -m' \end{pmatrix}\Big[l^{1/2}\begin{pmatrix} l+1 & 1 & l \\ m & 0 & -m \end{pmatrix} C(R^{l+1,m}, I^{l',m'}, I^{l'',m''}) \\ &+ (l+1)^{1/2}\begin{pmatrix} l-1 & 1 & l \\ m & 0 & -m \end{pmatrix} \times C(R^{l-1,m}, I^{l',m'}, I^{l'',m''})\Big]\Big\}, \end{aligned} \tag{A7}$$

where

$$\begin{aligned} n_7(m) &= 2^{1/2} \quad \text{if } m = 0 \\ &= 1 \quad \text{if } m \geq 1\,. \end{aligned} \tag{A8}$$

The Wigner $3j$ symbols used in equations (A1)–(A8) are all of the special form

$$\begin{pmatrix} l & 1 & l'' \\ m & m' & m'' \end{pmatrix},$$

which has a simple analytic form that can be found in Messiah (1958). The equations in this appendix are well defined only for values of m, m', m'', l, l', and l'' that give nonzero values of B, i.e., $m = |m' \pm m''|$, and $|l' - l''| + 1 \leq l \leq l' + l'' - 1$.

The C and B numbers have the following relationships for m and m' not equal to zero:

$$B(R^{l'',m+m'}R^{l'm'}I^{l,m}) = B(R^{l'',m+m'}I^{l'm'}R^{lm}) = -B(I^{l'',m+m'}R^{l'm'}R^{lm}) = B(I^{l'',m+m'}I^{l'm'}I^{lm})\,, \tag{A9}$$

$$B(R^{l''|m-m'|}R^{l'm'}I^{lm}) = -B(R^{l''|m-m'|}I^{l'm'}R^{lm}) = \epsilon B(I^{l''|m-m'|}R^{l'm'}R^{lm}) = \epsilon B(I^{l''|m-m'|}I^{l'm'}I^{lm}) \quad \text{for } m \neq m'\,, \tag{A10}$$

$$C(R^{l''m+m'}R^{l'm'}R^{lm}) = -C(R^{l''m+m''}I^{l'm'}I^{lm}) = C(I^{l''m+m'}R^{l'm'}I^{lm}) = C(I^{l''m+m'}I^{l'm'}R^{lm})\,, \tag{A11}$$

$$C(R^{l''|m-m'|}R^{l'm'}R^{lm}) = C(R^{l''|m-m'|}I^{l'm'}I^{lm}) = -\epsilon C(I^{l''|m-m'|}R^{l'm'}I^{lm}) = \epsilon C(I^{l''|m-m'|}I^{l'm'}R^{lm}) \quad \text{for } m \neq m'\,, \tag{A12}$$

where

$$\begin{aligned} \epsilon = &\;\;\,1 \quad (m' > m) \\ &-1 \quad (m' < m)\,. \end{aligned} \tag{A13}$$

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