J. Fluid Mech. (1977), *vol.* 79, *part* 3, *pp.* 525–534
Printed in Great Britain

On Green's functions for small disturbances of plane Couette flow

By PHILIP S. MARCUS
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540

AND WILLIAM H. PRESS
Center for Astrophysics and Department of Physics, Harvard University, Cambridge, Massachusetts 02138

(Received 7 June 1976 and in revised form 1 October 1976)

The linearized stability of plane Couette flow is investigated here, without using the Orr–Sommerfeld equation. Rather, an unusual symmetry of the problem is exploited to obtain a complete set of modes for perturbations of the unbounded (no walls) flow. An explicit Green's function is constructed from these modes. The unbounded flow is shown to be rigorously stable. The bounded case (with walls) is investigated by using a 'method of images' with the unbounded Green's function; the stability problem in this form reduces to an *algebraic* characteristic equation (not a differential-equation eigenvalue problem), involving transcendental functions defined by integral representations.

1. Introduction

There is not much doubt that viscous plane Couette flow is always stable to small disturbances, ones which satisfy the linear Orr–Sommerfeld perturbation equation. Nevertheless, no direct proof of stability is known. The evidence comes instead from asymptotic and numerical work which must be pieced together with some delicacy, as in Davey (1973). The sticking point of fully analytic investigations (Wasow 1953; Grohne 1954; Joseph 1968, and others) is the variety of different asymptotic regimes in the α (wavenumber of disturbance), R (Reynolds number) plane. Numerical investigations (Gallagher & Mercer 1962, 1964; Deardorff 1963, and others) have faced the unfortunate sensitivity of the Orr–Sommerfeld equation at high Reynolds number to various sorts of truncation error [as Orszag (1971) and Hughes (1972) have elaborated in different contexts].

Since the Orr–Sommerfeld equation is so uncooperative, it would seem reasonable to investigate how far one can proceed *without it*. This is the point of view that we take in this paper, and the answer that we find is: surprisingly far. Exploiting the symmetries of the background flow in a way that the Orr–Sommerfeld equation does not, we are able to obtain a complete set of modes for perturbations of unbounded plane Couette flow, where the walls are removed to infinity (§2). These modes are superposed in §3 to get an explicit Green's function for the space and time disturbances of an initial disturbance, and we show that in this unbounded case the flow is rigorously stable to linear disturbances at all wave-numbers. Returning to the wall-bounded case in §4, we impose boundary conditions on our Green's functions by something like a 'method of images', and we thereby obtain an algebraic transcendental equation for the complex eigen-frequencies of unstable modes. This algebraic equation thus contains the same stability information as does the differential-equation eigenvalue problem which comes from the Orr–Sommerfeld equation, which has thus been entirely circumvented. The functions which appear in our algebraic equation are rather complicated Laplace transforms of functions of complementary error functions, from which the supposed stability of the system is not immediately apparent. The equation does, however, elucidate the physical nature of an instability should one be present.

The symmetry which leads to the vorticity solution of equation (10) below was first recognized by Lord Kelvin (Thomson 1887) some two decades before Orr's (1907) classic work. Kelvin purported to use (10) to prove that plane Couette flow is stable. In fact, the proof contains flaws; Orr correctly recognized these, and offered his own method, since become standard, as a way of circumventing the Kelvin analysis. This paper, by contrast, picks up Kelvin's cold trail and proceeds further with it†. (We shall note Kelvin's errors below.)

2. Perturbation modes for unbounded flow

When there are no bounding walls, the unperturbed flow $\mathbf{V}_0$ is taken to have a velocity profile

$$V_{0x} = \sigma y, \quad V_{0y} = 0, \tag{1}$$

where σ is any constant. Adopting standard techniques for two-dimensional flow of an incompressible viscous fluid (Lin 1945, 1955), we let

$$V_x = \partial\Psi(x,y,t)/\partial y, \quad V_y = -\partial\Psi(x,y,t)/\partial x, \tag{2}$$

so that

$$\omega \equiv (\nabla \times \mathbf{V})_z = -\nabla^2\Psi, \tag{3}$$

where ω is the vorticity and Ψ is the stream function. The vorticity Navier–Stokes equation (e.g. Townsend 1956, chap. 1) is

$$(\nabla^2\Psi)_{,t} + \Psi_{,y}(\nabla^2\Psi)_{,x} - \Psi_{,x}(\nabla^2\Psi)_{,y} = \nu\nabla^4\Psi, \tag{4}$$

where ν is the kinematic viscosity.

Now setting $\Psi = \Psi_0 + \Psi_1$ (where Ψ_0 is the background flow and Ψ_1 a small perturbation) and linearizing, we obtain from (1)–(4) the equation for ω_1 $(\equiv -\nabla^2\Psi_1)$:

$$\omega_{1,t} + \sigma y \omega_{1,x} - \nu\nabla^2\omega_1 = 0. \tag{5}$$

Because t and x do not appear explicitly in (5), Lin (1955, p. 28) and others back to Orr and Sommerfeld chose to separate variables by

$$\Psi_1 = \phi(y)\exp(ikx - i\omega t), \tag{6a}$$

† We became aware of this trail only after our work had been completed, and we thank C. C. Lin for directing us to the historical literature.

which implies

$$\omega_1 = -(\phi'' - k^2\phi)\exp(ikx - i\omega t) \tag{6b}$$

(where a prime denotes d/dy), and obtained

$$(k\sigma y - \omega)(\phi'' - k^2\phi) = -i\nu(\phi^{\mathrm{iv}} - 2k^2\phi'' + k^4\phi), \tag{7}$$

which is the Orr–Sommerfeld equation for this flow. That this separation exists is a direct consequence of the translational symmetries of the background flow in the x and t co-ordinates. However, there is also a symmetry associated with the y direction: a combination of y translation and an x-velocity boost to a moving frame, or (equivalently) a pure y_L translation in the *Lagrangian* co-ordinates

$$y_L \equiv y, \quad x_L \equiv x - y\sigma t. \tag{8}$$

So an alternative ansatz to (6) is the separation of variables

$$\omega_1(x,y,t) = g(t)\exp(ikx_L + ily_L) = g(t)\exp[ikx + i(l - \sigma kt)y]. \tag{9}$$

Substitution into (5) gives an ordinary differential equation for $g(t)$. But in contrast to equation (7) for $\phi(y)$, this one is soluble by inspection, giving (with arbitrary constant of integration G)

$$\omega_1 = G\exp\{ikx + i(l - \sigma kt)y - \nu t[k^2 + \tfrac{1}{4}l^2 + \tfrac{1}{3}(k\sigma t - \tfrac{3}{2}l)^2]\}. \tag{10}$$

This solution was first obtained by Kelvin (Thomson 1877). A related separation of variables has been used more recently by Goldreich & Lynden-Bell (1965) in treating differential rotation in a gaseous disk. The modes (10) with different k's and l's are manifestly a complete set of solutions, because at time $t = 0$ they reduce to $\omega_1 \propto \exp(ikx + ily)$, a complete set of spatial Fourier components which can represent any initial vorticity perturbation.

One sees already in (10) that stability is likely, since at late times

$$\omega_1 \propto \exp(-\tfrac{1}{3}\sigma^2\nu k^2 t^3).$$

We pursue this point in the next section. For reference let us give here the stream function and perturbed velocities which correspond to the vorticity mode of (10):

$$\Psi_1 = \frac{-\omega_1}{k^2 + (l - \sigma kt)^2} + \mathscr{L}(x,y,t), \tag{11}$$

$$V_{1x} = \frac{-i\omega_1(l - \sigma kt)}{k^2 + (l - \sigma kt)^2} + \frac{\partial}{\partial y}\mathscr{L}(x,y,t), \tag{12}$$

$$V_{1y} = \frac{ik\omega_1}{k^2 + (l - \sigma kt)^2} - \frac{\partial}{\partial x}\mathscr{L}(x,y,t), \tag{13}$$

where $\mathscr{L}$ is any function with $\nabla^2\mathscr{L} = 0$. The terms in $\mathscr{L}$ arise from the incompressible potential flow that may always be added to a solution of the vorticity equation. If the perturbation velocity is required to be regular at all spatial infinity, then we must have $\mathscr{L} = 0$.

3. Stability of the unbounded flow: Green's functions

Since every spatial Fourier component of an initial disturbance dies out as $\exp(-\text{constant}\times t^3)$ at late times, an arbitrary perturbation will die with time in some mean-square sense. To prove linearized stability we need more than this (and at this point we diverge from Kelvin's treatment): we must show that the velocity goes *uniformly* to zero at all points in space, and moreover is bounded in magnitude at all finite times. These stronger conditions will exclude the possibility (otherwise left open) of the different Fourier components interfering constructively to produce anomalous points where the velocity might not go to zero, and might even become singular (vitiating the assumption of linearized flow).

The first step is to convert the perturbation modes of (10) into Green's functions. A straightforward Fourier transform of (10) gives the following result: vorticity whose value at $t=0$ is $\delta(x-x_0)\,\delta(y-y_0)$ has the complete time development

$$\omega_1(x,y,t) = \frac{1}{4\pi\nu t}(1+\tfrac{1}{12}\sigma^2t^2)^{-\frac{1}{2}} \times \exp\frac{-(x-x_0-\sigma y_0 t)^2+(x-x_0-\sigma y_0 t)(y-y_0)\,\sigma t-(y-y_0)^2(1+\frac{1}{3}\sigma^2t^2)}{4\nu t(1+\frac{1}{12}\sigma^2t^2)}. \tag{14}$$

One sees that this vorticity has the familiar form of a spreading Gaussian packet, as in a diffusion process, but modified by (i) a shearing of shape in the x, y plane and (ii) new time-dependent terms on σ^2t^2. Viewed from a co-ordinate system (x', y') whose centre is the comoving fluid point $(x,y) = (x_0+\sigma y_0 t, y_0)$ and which is rotated by an angle $\theta = \frac{1}{2}\arctan(3/\sigma t)$ with respect to the x, y co-ordinates, (14) takes the form

$$\omega_1(x',y',t) = h(t)\exp\{-p(t)\,[x'^2+(1-e^2)\,y'^2]\}, \tag{15}$$

where

$$h(t) = (4\pi\nu t)^{-1}(1+\tfrac{1}{12}\sigma^2t^2)^{-\frac{1}{2}}, \tag{16}$$

$$p(t) = \tfrac{1}{8}(\nu t+\tfrac{1}{12}\sigma^2\nu t^3)^{-2}[2+\tfrac{1}{3}\sigma^2t^2+(3+\tfrac{1}{3}\sigma^2t^2)(1+9/\sigma^2t^2)^{-\frac{1}{2}}], \tag{17}$$

$$1-e^2 = \frac{2+\frac{1}{3}\sigma^2t^2-(1+9/\sigma^2t^2)^{-\frac{1}{2}}(3+\frac{1}{3}\sigma^2t^2)}{2+\frac{1}{3}\sigma^2t^2+(1+9/\sigma^2t^2)^{-\frac{1}{2}}(3+\frac{1}{3}\sigma^2t^2)}. \tag{18}$$

The functions $h(t)$, $p(t)$ and $e(t)$ are all monotonic functions of time.

The second step is to consider the perturbed velocity field derived from the vorticity field $\omega_1(x',y',t)$ in (15), namely

$$V_{1y'}(x',y',t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}d\xi\int_{-\infty}^{+\infty}d\eta\,\frac{x'-\xi}{(x'-\xi)^2+(y'-\eta)^2}\,\omega_1(\xi,\eta,t), \tag{19}$$

$$V_{1x'}(x',y',t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}d\xi\int_{-\infty}^{+\infty}d\eta\,\frac{\eta-y'}{(x'-\xi)^2+(y'-\eta)^2}\,\omega_1(\xi,\eta,t). \tag{20}$$

We now want to find bounds on $V_{1y'}$ and $V_{1x'}$, without evaluating the integrals (19) and (20), which are not elementary. To do this, introduce a 'trial' vorticity field ω_T, given by

$$\omega_T(x',y',t) = h(t)\exp[-p(t)\,x'^2]. \tag{21}$$

Equation (19) with ω_T substituted for ω_1 yields

$$V_{Ty'}(x'\,y'\,t) = \tfrac{1}{4}[\pi/p(t)]^{\frac{1}{2}}h(t)\{\operatorname{erfc}[-p(t)^{\frac{1}{2}}x'] - \operatorname{erfc}[p(t)^{\frac{1}{2}}x']\}, \tag{22}$$

which is bounded by

$$|V_{Ty'}(x',y',t)| \leqslant \tfrac{1}{2}\pi^{\frac{1}{2}}h(t)/p(t)^{\frac{1}{2}}. \tag{23}$$

The relation between $V_{Ty'}$ [bounded according to (23)] and $V_{1y'}$ (our perturbation velocity) is now demonstrated. From (19) and (15), some algebra gives

$$V_{1y'} = \frac{h}{\pi}\int_{-\infty}^{+\infty} d\eta \exp[-(1-e^2)\eta^2 p]\int_0^\infty d\xi \frac{\xi}{\xi^2+(y'-\eta)^2} \times \exp[-p(x'^2+\xi^2)]\sinh(2px'\xi). \tag{24}$$

This integral is positive (negative) definite for $x' > 0$ ($x' < 0$), so $V_{1y} \geqslant 0$ ($V_{1y} \leqslant 0$) for $x' > 0$ ($x' < 0$). Now we also can find the sign of $V_{Ty'} - V_{1y'}$:

$$V_{Ty'}(x'\,y',t) - V_{1y'}(x',y',t) = \frac{h}{2\pi}\int_{-\infty}^{+\infty} d\eta\{1-\exp[-(1-e^2)\eta^2 p]\}\int_0^\infty d\xi\frac{\xi}{\xi^2+(y'-\eta)^2} \times \exp[-p(\xi^2+x'^2)]\sinh(2p\xi x'). \tag{25}$$

Again the integral is positive (negative) definite for $x' > 0$ ($x' < 0$). This proves [using (23)] that

$$|V_{1y'}| \leqslant |V_{Ty'}| \leqslant \tfrac{1}{2}\pi^{\frac{1}{2}}h/p^{\frac{1}{2}}. \tag{26}$$

Now for the other component $V_{1x'}$, one performs exactly the same analysis (22)–(26) but starting with a different ω_T,

$$\omega_T(x',y',t) = h\exp[-p(1-e^2)y'], \tag{27}$$

and obtains

$$|V_{1x'}| \leqslant |V_{Tx'}| \leqslant \tfrac{1}{2}\pi^{\frac{1}{2}}h/p^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}}. \tag{28}$$

Combining (27), (28) and $0 \leqslant 1-e^2 \leqslant 1$, we have our uniform bound for the perturbation velocity:

$$V_1(x',y',t) \leqslant (\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{h}{p^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}}} \leqslant \frac{1}{(2\pi\nu t)^{\frac{1}{2}}}. \tag{29}$$

Recalling that this velocity field came originally from a delta-function vorticity perturbation [see (14)], we are now able to state our most general theorem for spatially unbounded plane Couette flow.

THEOREM. If at a time $t = 0$ a velocity perturbation is confined to a bounded spatial region, and if the first spatial derivatives of the velocity perturbation are bounded, then in its subsequent evolution the velocity perturbation is uniformly bounded over all space, and goes to zero in time uniformly, at least as constant $\times(\nu t)^{-\frac{1}{2}}$.

Proof. The initial vorticity distribution is also bounded in amplitude and space, so the vorticity Green's function is immediately applicable and (29) applies.

In short, plane Couette flow without walls is admirably stable.

4. Stability of wall-bounded Couette flow

We consider now the effect of inserting bounding walls at $y = \pm h$, comoving with the unperturbed fluid at velocities $V_x = \pm \sigma h$, $V_y = V_z = 0$. The walls impose the new boundary conditions $V_{1x} = V_{1y} = 0$ at $y = \pm h$, for all x and t. The Green's function of (14) is not, therefore, a solution to the vorticity equation *plus boundary conditions*, even though it *is* a solution everywhere in the interior region $-h < y < h$. To satisfy the boundary conditions we are free, however, to add new, fictitious sources of vorticity anywhere except in the interior region. These are the analogue of the familiar 'image' sources in electromagnetism which allow free-space Green's functions to be used in the presence of some conducting surfaces. In the present problem, the image sources of vorticity may be restricted to lie exactly on the walls $y = \pm h$, as we now show. Without loss of generality we Fourier analyse all quantities in the x direction and take their x dependence as exp ikx. At some instant of time $t = t'$ the perturbation velocity field will be a sum of the Green's functions of the initial ($t = 0$) perturbation and the Green's functions of previous image vorticity on the walls, i.e. for all $t < t'$. We want to show that whatever velocity these produce at $y = \pm h$ can now be exactly cancelled by a suitable choice of image vorticity at $t = t'$, $y = \pm h$. Let the velocity to be cancelled by $v^G e^{ikx}$, and let the image vorticity at $t = t'$ be

$$\omega_i(x, y) = [C_1 \delta(y-h) + C_2 \delta(y+h)] e^{ikx}. \tag{30}$$

This vorticity produces a velocity (still at time t')

$$v_y^i(x, h) = -\frac{i}{2}\frac{|k|}{k} e^{ikx}(C_1 + C_2 e^{-2h|k|}), \tag{31a}$$

$$v_y^i(x, -h) = -\frac{i}{2}\frac{|k|}{k} e^{ikx}(C_1 e^{-2h|k|} + C_2), \tag{31b}$$

$$v_x^i(x, h) = v_x^i(x, -h) = 0. \tag{31c}$$

There can also be superposed a velocity due to potential flow [see (12) and (13)], since we do not require that the solution be regular as $y \to \infty$ (outside the walls). The most general potential $\mathscr{L}$ with e^{ikx} dependence is

$$\mathscr{L} = e^{ikx}(C_3 e^{ky} + C_4 e^{-ky}). \tag{32}$$

From (12), (13), (31) and (32) the boundary condition is now seen to be a linear system of equations for C_1, C_2, C_3 and C_4, namely

$$\begin{pmatrix} 0 & 0 & e^{kh} & -e^{-kh} \\ 0 & 0 & e^{-kh} & -e^{kh} \\ \frac{1}{2}|k|^{-1} & \frac{1}{2}|k|^{-1}e^{-2|k|h} & e^{kh} & e^{-kh} \\ \frac{1}{2}|k|^{-1}e^{-2|k|h} & \frac{1}{2}|k|^{-1} & e^{-kh} & e^{kh} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = -\frac{1}{k} \begin{pmatrix} v_x^G(h) \\ v_x^G(-h) \\ iv_y^G(h) \\ iv_y^G(-h) \end{pmatrix}. \tag{33}$$

It is readily shown that the matrix in (33) is never singular for any finite h, so there is always a unique solution to (33). We have thus shown that the general time-dependent solution for a perturbation of wall-bounded Couette flow is given by (i) the Green's function development (14) of the initial disturbance plus (ii) the Green's function development of a uniquely determined [by (33)] vorticity source at the walls plus (iii) a uniquely determined [by (33)] potential flow which varies with time.

We turn now to the question of stability. Can an initial velocity (hence vorticity) perturbation give rise to a solution which grows with time at late times? Let the vorticity perturbation $e^{ikx}\delta(y-y'')$ be introduced at time t''; since k and y'' are arbitrary, this perturbation is representative of a complete set of perturbations. The Green's function for this source at all times $t > t''$ is [by a Fourier transformation in x of (14) or in y of (10)]

$$G_k(x,y,t,y'',t'') = \tfrac{1}{2}[\pi\nu(t-t'')]^{-\frac{1}{2}} \exp[ikx - (y-y'')^2/4\nu(t-t'') - \tfrac{1}{12}\sigma^2k^2\nu \times (t-t'')^3 - \nu k^2(t-t'') - \tfrac{1}{2}i(y+y'')\sigma k(t-t'')]. \quad (34)$$

Notice that at late times this decays as $\exp(-\text{constant} \times t^3)$. However, we know that image vorticity is created at the walls to satisfy the boundary conditions. Any given image source decays with time as in (34), but while it is decaying it induces *new* image vorticity in the walls. The creation of image vorticity is thus a continuous process, and the flow will be unstable only if this creation occurs 'faster' than the decay of vorticity. As Orr pointed out, Kelvin seems to have erred principally on this point: since G [or ω, in (10)] decays with time, Kelvin imagined without justification that the response of the walls must also decay in time, and he thus felt justified in Fourier transforming the response of the wall using real frequencies only. But this is tantamount to *assuming* the stability that he finally claims to prove! We avoid this pitfall. To represent this quantitatively, let the image vorticity source at $y = \pm h$ be $F_1(t)\, e^{ikx}\,\delta(y-h) + F_2(t)\, e^{ikx}\delta(y+h)$, and let the potential flow (also required by boundary conditions) be generated from the potential

$$\mathscr{L} = e^{ikx}[F_3(t)\sinh(ky) + F_4(t)\cosh(ky)]. \quad (35)$$

The total vorticity at time t due to all previous image sources is

$$\omega_i = \int_{-\infty}^{t} [F_1(t')\, G_k(x,y,t,h,t') + F_2(t')\, G_k(x,y,t,-h,t')]\, dt', \quad (36)$$

where G_k is the Green's function in (34). A tedious but straightforward calculation gives the velocity field $\mathbf{v}^i$ which corresponds to this vorticity field:

$$\left.\begin{aligned} v_x^i(x,y,t) &= e^{ikx}[U_x(F_1,t,y,\alpha) + U_x(F_2,t,y,-\alpha)],\\ v_y^i(x,y,t) &= e^{ikx}[U_y(F_1,t,y,\alpha) + U_y(F_2,t,y,-\alpha)],\end{aligned}\right\} \quad (37)$$

where

$$U_x(F_1,t,y,\alpha) \equiv \frac{R}{4\sigma\alpha^2}\int_0^\infty dt' F_i\left(t - \frac{t'}{\nu k^2}\right)\exp\left(\frac{-R^2t'^3}{3\alpha^4} \mp i\frac{Rt'}{\alpha}\right) \times \left[\operatorname{erfc}\left(\frac{|k|y-\alpha}{2t'^{\frac{1}{2}}} \pm \frac{iRt'^{\frac{3}{2}}}{2\alpha^2} + t'^{\frac{1}{2}}\right)\exp\left(|k|y \pm i\frac{Rt'^2}{\alpha^2} - \alpha\right) - \operatorname{erfc}\left(\frac{\alpha-|k|y}{2t'^{\frac{1}{2}}} \mp i\frac{Rt'^{\frac{3}{2}}}{2\alpha^2} + t'^{\frac{1}{2}}\right)\exp\left(-|k|y \mp \frac{iRt'^2}{\alpha^2} + \alpha\right)\right], \quad (38)$$

$$U_y(F_i,t,y,\alpha) \equiv \mp \frac{iR}{4\sigma\alpha^2}\int_0^\infty dt'\, F_i\left(t-\frac{t'}{\nu k^2}\right)\exp\left(-\frac{R^2t'^3}{3\alpha^4}\mp\frac{iRt'}{\alpha}\right)$$
$$\times\left[\operatorname{erfc}\left(\frac{|k|y-\alpha}{2t'^{\frac{1}{2}}}\pm\frac{iRt'^{\frac{3}{2}}}{2\alpha^2}+t'^{\frac{1}{2}}\right)\exp\left(|k|y\pm\frac{iRt'^2}{\alpha^2}-\alpha\right)\right.$$
$$\left.+\operatorname{erfc}\left(\frac{\alpha-|k|y}{2t'^{\frac{1}{2}}}\mp\frac{iRt'^{\frac{3}{2}}}{2\alpha^2}+t'^{\frac{1}{2}}\right)\exp\left(-|k|y\mp\frac{iRt'^2}{\alpha^2}+\alpha\right)\right]. \quad (39)$$

Here we have introduced the dimensionless constants

$$R \equiv \sigma h^2/\nu \quad \text{(Reynolds number)} \quad (40)$$

and $\alpha \equiv |k|\,h$. The upper (lower) signs in (38) and (39) refer to $k > 0\,(k < 0)$. If $\mathbf{V}_G(y,t)\,e^{ikx}$ is the velocity perturbation whose vorticity is (34), then the four equations which impose the wall boundary conditions are

$$U_x(F_1,t,\alpha,\alpha)+U_x(F_2,t,\alpha,-\alpha)+k(F_3\cosh\alpha\pm F_4\sinh\alpha) = -V_{Gx}(h,t), \quad (41)$$

$$U_x(F_1,t,-\alpha,\alpha)+U_x(F_2,t,-\alpha,-\alpha)+k(F_3\cosh\alpha\mp F_4\sinh\alpha) = -V_{Gx}(-h,t), \quad (42)$$

$$U_y(F_1,t,\alpha,\alpha)+U_y(F_2,t,\alpha,-\alpha)-ik(\pm F_3\sinh\alpha+F_4\cosh\alpha) = -V_{Gy}(h,t), \quad (43)$$

$$U_y(F_1,t,-\alpha,\alpha)+U_y(F_2,t,-\alpha,-\alpha)-ik(\mp F_3\sinh\alpha+F_4\cosh\alpha) = V_{Gy}(-h,t). \quad (44)$$

Now we come to an important physical point. If there is an instability in the flow, then the individual terms on the left-hand sides of (41)–(44) should be growing in time. However, the terms on the right-hand sides represent the Green's function of the original perturbation, which *decays* exponentially in time. So at very late times the right-hand sides are negligibly small; for a *pure* growing mode (one which has been growing for all time), the right-hand sides must be effectively zero. But in this case the left-hand sides do not depend explicitly on t at all, except through the arguments of the F_i, which enter linearly. Therefore, at this stage we can recognize that the linear operator which acts on the F_i is invariant under time translation, and that we should adopt the ansatz

$$F_i(t) = C_i\,e^{\lambda t} \quad (i = 1,2,3,4), \quad (45)$$

i.e. we can just look for exponentially growing modes with $\operatorname{Re}\lambda \geqslant 0$. We could *not* do this before setting the right-hand sides to zero, because the Green's function for the initial disturbance is not a simple exponential in time. The point is that, with a growing mode, the time of the initial disturbance can be pulled back to negative infinity, i.e. $t'' \to -\infty$. The physical nature of any instability must then be a 'resonance' in vorticity production on the two walls, and this problem (with t now an ignorable co-ordinate) should have exponential solutions.

Does such a resonance exist? In other words, do any values of λ such that $\operatorname{Re}\lambda \geqslant 0$ satisfy (41)–(45) for some complex constants C_i ($i = 1,\ldots,4$) and with the right-hand sides of (41)–(44) zero? The equations are linear and homogeneous in the C_i. It is straightforward to eliminate C_3 and C_4 from the equations, and obtain two equations for C_1 and C_2. After some manipulation, one is led to define two transcendental functions in terms of Laplace transforms of exponential and complementary error functions of complicated arguments:

$$P_1(\lambda, R, \alpha) \equiv \int_0^\infty e^{-\lambda z} \exp\left(-\frac{R^2 z^3}{3\alpha^4} - \frac{iRz^2}{\alpha}\right) \operatorname{erfc}\left(\frac{iRz^{\frac{3}{2}}}{2\alpha^2} + z^{\frac{1}{2}}\right) dz, \quad (46)$$

$$P_2(\lambda, R, \alpha) \equiv e^{-2\alpha^2} \int_0^\infty e^{-\lambda z} \exp\left(-\frac{R^2 z^3}{3\alpha^4} \mp \frac{iRz^2}{\alpha}\right) \operatorname{erfc}\left(\frac{iRz^{\frac{3}{2}}}{2\alpha^2} + z^{\frac{1}{2}} - \frac{\alpha}{z}\right) dz. \quad (47)$$

In terms of these functions, the condition that solutions for C_1 and C_2 exist takes the form of a remarkably simple 2×2 determinant equation:

$$0 = \begin{vmatrix} \tanh\alpha[P_1(\lambda, R, \alpha) + P_2(\lambda, R, \alpha)] - [P_1(\lambda, R, \alpha) - P_2(\lambda, R, \alpha)] & \tanh\alpha[P_1(\lambda, R, -\alpha) + P_2(\lambda, R, -\alpha)] + [P_1(\lambda, R, -\alpha) - P_2(\lambda, R, -\alpha)] \\ -\tanh\alpha[P_1^*(\lambda, R, -\alpha) + P_2^*(\lambda^*, R, -\alpha)] - [P_1^*(\lambda^*, R, -\alpha) - P_2^*(\lambda^*, R, -\alpha)] & -\tanh\alpha[P_1^*(\lambda^*, R, \alpha) + P_2^*(\lambda^*, R, \alpha)] + [P_1^*(\lambda^*, R, \alpha) - P_2^*(\lambda^*, R, \alpha)] \end{vmatrix}. \quad (48)$$

Here an asterisk signifies complex conjugation. Equation (48) is a transcendental equation in R (Reynolds number), α (dimensionless wavenumber) and λ (a complex frequency). For fixed R and α, it is a transcendental equation for λ; any instability of the flow will appear as a root with $\operatorname{Re} \lambda > 0$.

5. Discussion

We have seen that unbounded plane Couette flow is stable to small disturbances, and that the unstable modes of wall-bounded plane Couette flow (if they exist) have eigenfrequencies which are solutions to the transcendental equation (48). These growing modes must also be eigensolutions of the Orr–Sommerfeld equation (7), which is a fourth-order differential equation with two-point boundary conditions, but we have completely circumvented that equation in the treatment here. Indeed, the only ordinary differential equation which appeared here was of first order in time, and soluble by inspection [giving (10)].

Of course, the functions which appear in (48) are not elementary. One might ask, why not simply *define* four functions to be the independent solutions of the Orr–Sommerfeld equation and thus reduce the standard approach to a formally 'algebraic' problem? The formal difference between this and our treatment is that the functions in (48) are given as new, explicit integrals [see (46) and (47)]. For the Orr–Sommerfeld equation, (5) can be solved in terms of Airy functions (Orr 1907), so a formal solution to (7) in terms of integrals of these is possible; but this approach does not seem to have been fruitful. The integrals of this paper look to be quite tractable for numerical evaluation: focusing interest on the *onset* of possible instability, one would search for purely imaginary roots λ, so the Laplace transforms could be evaluated by fast Fourier transform techniques. The functions inside the integral could also be evaluated efficiently; for example, Abramowitz & Stegun (1970, p. 328) give concise approximations for erfc for arbitrary complex arguments. We have not, however, pursued these numerical questions. On the analytic side, one might hope that an analytic proof of stability could come from the study of the functions (46) and (47) in the complex plane.

Some of the results of this paper generalize to perturbations of any viscous shear flow whose unperturbed velocities are linear functions of the co-ordinates. The translational symmetry in Lagrangian co-ordinates provides the opening wedge in these cases, as in plane Couette flow.

Our work on this subject has benefited from discussions of a general nature with S. Chandrasekhar and A. Toomre. We thank C. C. Lin for comments on the manuscript. W.H.P. thanks the Institute of Astronomy, University of Cambridge for its hospitality during the early stages of this work. We thank the U.S. National Science Foundation for partial support under grants GP 30799X and PHY 76-14852.

REFERENCES

ABRAMOWITZ, M. & STEGUN, I. A. 1970 *Handbook of Mathematical Functions*. Washington: Nat. Bur. Stand.

DAVEY, A. 1973 *J. Fluid Mech.* **57**, 369.

DEARDORFF, J. W. 1963 *J. Fluid Mech.* **15**, 623.

GALLAGHER, A. P. & MERCER, A. McD. 1962 *J. Fluid Mech.* **13**, 91.

GALLAGHER, A. P. & MERCER, A. McD. 1964 *J. Fluid Mech.* **18**, 350.

GOLDREICH, P. & LYNDEN-BELL, D. 1965 *Mon. Not. Roy. Astr. Soc.* **130**, 125.

GROHNE, D. 1954 *Z. angew. Math. Mech.* **34**, 344.

HUGHES, T. H. 1972 *Phys. Fluids*, **15**, 725.

JOSEPH, D. D. 1968 *J. Fluid Mech.* **33**, 617.

LIN, C. C. 1945 *Quart. Appl. Math.* **3**, 117.

LIN, C. C. 1955 *The Theory of Hydrodynamic Stability*. Cambridge University Press.

ORR, W. M. F. 1907 *Proc. Roy. Irish Acad.* A **27**, 69.

ORSZAG, S. A. 1971 *J. Fluid Mech.* **49**, 75.

THOMSON, W. 1887 *Phil. Mag.* **24** (5), 188.

TOWNSEND, A. A. 1956 *The Structure of Turbulent Shear Flow*. Cambridge University Press.

WASOW, W. 1953 *J. Res. Nat. Bur. Stand.* **51**, 195.