Nuclear Physics B (Proc. Suppl.) 2 (1987) 127-138
North-Holland, Amsterdam

SPATIAL SELF-ORGANIZATION OF VORTICITY IN CHAOTIC SHEARING FLOWS

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We examine two-dimensional incompressible fluid flows that are chaotic in time but have large scale coherent spatial structures. We show that spots of vorticity superposed on a chaotic shear flow can be robust and merge together if the shear and the vorticity have the same sign. We propose the hypothesis that the merging and stability criteria for spots can be derived by requiring that the energy in the time-averaged (non-chaotic) component of the velocity is minimized. Our numerical simulations of the dynamics of these spots is qualitatively similar to that of the Red Spot of Jupiter.

1. INTRODUCTION

In this paper we report numerical calculations of a fluid flow in which spatial structure and large-scale self-organization coexist with temporal chaos. We consider the incompressible flow between two concentric cylinders of radii R_1 and R_2. The upper boundary of the annulus is flat and located at height $z = H$, and the lower boundary is radially sloped with position $z=-r \cdot s$. The annulus is rotated about the z-axis with angular velocity Ω. All equations and results reported in this paper are in this rotating frame. If $\langle\omega\rangle/\Omega \equiv Ro \ll 1$ then the Taylor-Proudman theorem requires that to first order in Ro the flow is two-dimensional with no velocity component and no velocity gradient in the axial z direction. Here, $\langle\omega\rangle$ is the characteristic z-component of the vorticity of the fluid in the rotating frame and Ro is the Rossby number. If in addition, the slope is small $|s \cdot (R_2-R_1)/H| \ll 1$ the three-dimensional Euler and continuity equations and boundary conditions have the two-dimensional (quasi-geostropic) approximations[1]

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla)\underline{u} = \frac{D\underline{u}}{Dt} = \beta r(\hat{e}_z \times \underline{u}) - \nabla \Pi \qquad (1)$$

$$\nabla \cdot \underline{u} = 0 \qquad (2)$$

where Π is the pressure head of the fluid, $\beta \equiv 2\Omega s/H$ is a dimensionless constant, and $D/Dt \equiv \partial/\partial t + \underline{u} \cdot \nabla$ is the advective derivative. We have non-dimensionalized equation (1) with unit of length equal to (R_2-R_1), time equal to $1/s\ \Omega$, and mass density of the fluid equal to unity. In equation (1) we have ignored viscosity. Its role in the dynamics discussed in this paper is unimportant,

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(North-Holland Physics Publishing Division)

although it is included in our numerical calculations when we make comparisons with laboratory experiments[2]. The boundary conditions for the inviscid flow are

$$u_r = 0 \quad \text{at} \quad r = R_1 \text{ and } R_2 \tag{3}$$

The physics is more intuitive by taking the curl of equation (1) which describes the vorticity dynamics ($\omega \equiv \hat{e}_z \cdot \nabla \times \underline{u}$)

$$\frac{D\omega}{Dt} = \beta u_r \tag{4}$$

or

$$\frac{D\omega_p}{Dt} = 0 \tag{5}$$

where $\omega_p \equiv \omega - \beta r$ is defined to be the potential vorticity. Equation (4) states that in the absence of a sloping bottom the vorticity of the flow is advected along with each fluid element ($D\omega/Dt = 0$). The inclined bottom boundary stretches each vortex filament, and the stretch causes the vorticity carried by a fluid element to change. The advective change of the vorticity is proportional to the slope of the bottom and the speed that the fluid element goes from deep to shallow water. Equation (5) shows that the potential vorticity of the flow is unchanged as it is advected.

One solution of equation (5) is $\omega_p = C_1$ where C_1 is a constant. This is an important solution because a random stirring of the fluid (caused by instabilities or external forcing) can cause an ergodic mixing of fluid elements and hence of potential vorticity. The formation of flows with large-scale or macroscopically uniform ω_p (and with miscroscopic fluctuations in ω_p) was predicated from numerical experiments by Marcus[2] and observed experimentally by Sommeria and Swinney[3]. The axi-symmetric velocity fields corresponding to $\omega_p = C_1$ are

$$\tilde{u}(r)\hat{e}_\phi = \left(\frac{\beta r^2}{3} + rC_1/2 + C_2/r\right)\hat{e}_\phi \tag{6}$$

where C_2 is also a constant. Notice that the shear of $\tilde{u}$ $\sigma(r) = \beta r/3 - 2C_2/r^2$ and is a function of position. It is convenient to think of the vorticity as the sum of two components, $(C_1 + \beta r)$ and $\omega_e(r,\phi,t)$ where ω_e is the vorticity in excess of the uniform potential vorticity flow

$$\omega_e(r,\phi,t) \equiv \omega(r,\phi,t) - (C_1 + \beta r) \tag{7}$$

Similarly we decompose the velocity into $\tilde{u}$ and its excess $\underline{u}_e$

$$\underline{u}_e(r,\phi,t) \equiv \underline{u}(r,\phi,t) - \tilde{u}(r)\hat{e}_\phi \tag{8}$$

where

$$\omega_e = (\nabla \times \underline{u}_e) \cdot \hat{e}_z \tag{9}$$

The excess vorticity (not to be thought of as a perturbation because the flows of interest in this paper have ω_e of order unity) is advected with the total velocity of the flow

$$\frac{D\omega_e}{Dt} = 0 \tag{10}$$

A fluid element advects its excess vorticity with no change in its strength — any spin up or down caused by stretching from the sloped bottom is exactly balanced by the spin down or up exerted by the torques from the gradient of the shear in $\tilde{u}$. Equation (10) shows that the excess circulation Γ

$$\Gamma \equiv \int_{R_1}^{R_2} rdr \int d\phi \; \omega_e \tag{11}$$

is conserved in time.

We note that adding a uniform rotation $r \cdot C\hat{e}_\phi$ to our inviscid, incompressible, two-dimensional velocity $\underline{u}$ in no way alters the physics of the flow because if $\underline{u}(r,\phi,t)$ is a solution to equations (1)-(3) then $\underline{u}(r,\phi - Ct, t) + rC\hat{e}_\phi$ is also a solution. Therefore without loss of generality we can set $C_1 = 0$, and we see that $\omega_e = \omega_p$.

In the remainder of this paper we show how the excess vorticity either self-organizes into one or more large coherent spots or fragments into thin chaotic filaments. Our motivation for looking for spatial self-organization in this particular type of annular flow comes from the coincidence that equation (1) is also the two-dimensional (quasi-geostrophic) approximation of a rapidly rotating planetary atmosphere. In the Jovian atmosphere the East-West zonal wind in which the Great Red Spot is located looks to a first approximation like a macroscopically axisymmetric uniform potential vorticity flow $\tilde{u}$. The Red Spot is a patch of excess vorticity.

2. SUMMARY OF NUMERICAL EXPERIMENTS

We have calculated numerical solutions to equations (1)-(3) using a de-aliased pseudospectral initial-value method with 256 Chebyshev and 256 Fourier modes in the radial and azimuthal directions. Details can be found in Marcus[4]. We summarize here the qualitative results of five different types of initial-value experiments. One feature common to all of our experiments is $R_1/R_2 = 0.25$.

2.1 One initial spot of negative excess vorticity

These experiments have an initial flow with $C_2 = 0$. This guarantees that the sign of $\sigma(r)$ does not change in the flow. (In fact, the characteristic radial distance over which $\sigma(r)$ varies is r, so σ changes only by a factor of 4 from R_1 to R_2.) The initial flow has one spot of excess vorticity, with ω_e approximately uniform throughout the spot. This set of experiments has $\langle\omega_e\rangle/\langle\sigma\rangle < 0$ where $\langle\omega_e\rangle$ is the characteristic ω_e of the spot and $\langle\sigma\rangle$ is the characteristic value of σ at the initial location of the spot. We therefore refer to these as negative spots. The initial values of the radial location, area, shape and $\langle\omega_e\rangle/\langle\sigma\rangle$ of the spots are changed in the experiments, but we restrict the initial areas of the spots to be less than or of order unity and the spots location to be not too close to a boundary.

The experiments show that when $|\langle\omega_e\rangle/\langle\sigma\rangle|$ is less than or of order unity the following occurs. The differential rotation of $\tilde{u}$ stretches the excess vorticity of the spot into a thin spiral; the excess vorticity initially located at large values of r is drawn to the outer boundary of the annulus and the material at small initial values of r to the inner boundary; the spiral fragments into thin filaments (each, by virtue of equation (10) retaining its initial value of ω_e); each filament is further stretched by the differential rotation into a new smaller, thinner spiral which then fragments into new filaments. (Note that fission and merger of spots is allowed kinematically in real laboratory experiments and in our numerical experiments because both are slightly dissipative.) The stretching and fragmentation occur on a quick time-scale — of order the spot turnaround time (defined to be $4\pi/\langle\omega_e\rangle$).

This fragmentation results in the initial excess vorticity accumulating in one of three places: at the inner boundary where an excess circulation Γ_1 is deposited, at the outer boundary where excess circulation Γ_2 is deposited, or dispersed throughout the annulus with an excess circulation of Γ_3. Conservation requires

$$\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 \qquad (12)$$

where Γ is the initial value of the excess circulation. The flow produced by Γ_3 is temporally chaotic and shows a broad band power spectrum. It is composed of excess-vorticity filaments that appear to be ergodically distributed such that the density of filaments (excess circulation/area) is approximately uniform throughout the entire annulus.

The values of Γ_1, Γ_2 and Γ_3 are established in approximately one turnaround time, and the subsequent fragmentation of excess vorticity into smaller filaments does not effect the values. The time-averaged component of $\underline{u}_e$ is

$$v(r)\hat{e}_\phi = [\Gamma_1/2\pi r + r\Gamma_3/2\pi(R_2^2-R_1^2)]\hat{e}_\phi \quad (13)$$

The piece of $\underline{u}_e$ due to the time-averaged filaments in Γ_3 is a uniform rotation. We define the instantaneous fluctuating excess velocity to be

$$\underline{v}_f(r,\phi,t) \equiv \underline{u}_e(r,\phi,t) - v(r)\hat{e}_\phi \quad (14)$$

Clearly, $\underline{v}_f$ has no circulation, and we shall argue below that it has no angular momentum.

We terminate our numerical simulations when the size of the filaments becomes smaller than the numerical resolution of our code. This allows us to simulate the flow for 15-30 turnaround times. There appears to be no macroscopic changes in the flow after the first few turnaround times when the values of Γ_1, Γ_2 and Γ_3 are established. We have continued the simulations past 30 turnaround times by adding a small numerical dissipation at the resolution lengthscale and still found that there are no macroscopic changes in the flow after 100 turnaround times.

When $|\langle\omega_e\rangle/\langle\sigma\rangle| \gg 1$ negative spots become long-lived. Not surprisingly, their behavior is similar to that of spots of excess vorticity with $\langle\sigma\rangle = \beta = s = C_2 = 0$ (i.e., the usual 2-dimensional inviscid vortex dynamics)[5].

2.2 One initial spot of positive excess vorticity

The initial conditions of these experiments were identical to those for one spot of negative excess vorticity with the exception that $\langle\omega_e\rangle/\langle\sigma\rangle > 0$. The spot is initially stretched in the azimuthal direction by $\tilde{u}$, then contracts. Repeated oscillations of stretch and contraction occur with decreasing amplitude. During these oscillations small pieces of the spot can be broken off and advected away by $\tilde{u}$, but if $\langle\omega_e\rangle/\langle\sigma\rangle$ is order unity, the loss of excess circulation from the spot is small and the area of the spot never decreases by more than 5%. After two or three turnaround times, the oscillations cease and the spot forms an elliptical shape with the long axis in the azimuthal direction. The ellipticity ε (ratio of the axis in the radial direction to that in the azimuthal) is observed to be approximately

$$\varepsilon \cong \omega_e/(\omega_e + \sigma) \quad (15)$$

The radial position of the positive spot is approximately the same as its initial value. The spot is advected azimuthally around the annulus by $\underline{u}(r,\phi,t)$ so the spot's speed is the characteristic azimuthal velocity at the radial location of the spot.

The fragments of ω_e lost by the spot during the initial oscillations are deposited at the boundaries or ergodically mixed throughout the annulus in a way that is qualitatively similar to the fragments of negative spots. Therefore, the final velocity is a superposition of a single, steady-state (in the proper rotating frame) elliptical positive spot, $\tilde{u}(r)$ from equation (6), v(r) from equation (13), and the chaotic component $\underline{v}_f(r,\phi,t)$.

Similar to negative spots, if a positive spot has $|\langle\omega_e\rangle/\langle\sigma\rangle| >> 1$ the spot behaves as an inviscid vortex with the usual 2-dimensional dynamics. Positive spots with $\langle\omega_e\rangle/\langle\sigma\rangle << 1$ have an ellipticity that is so small that they resemble shear layers, are unstable to Kelvin-Helmholtz instabilities, and break apart.

2.3 Multiple spots

When two or more negative spots are initially present, they all fragment into filaments. With one positive and one negative spot as the initial conditions, the negative spot still fragments and the positive spot still forms a nearly elliptical equilibrium. When two or more positive spots are present mergers are possible.

We have found with $C_2 = 0$, that if two positive spots are present initially with small impact parameter (radial separation) they merge as follows. If they are not too close in azimuthal location, they behave initially as isolated spots and form two nearly elliptical equilibria. They are advected towards each other azimuthally by the differential rotation of $\tilde{u}$. The speed at which they approach is therefore proportional to $\sigma(r)$ and to the impact parameter. (An impact parameter initially zero will not remain zero due to the tendency of spots to rotate about their center of excess vorticity.) When the separation between the spots becomes smaller than their characteristic size, they distort each other into spirals and wrap around each other (entraining fluid with no ω_e into them). The spots eject the fluid with no ω_e and form an elliptical equilibrium. The characteristic time for mergers is a few turnaround times. During the merging process some excess circulation is broken off the spots and forms thin filaments or is deposited at the boundaries, but in our experiments with $0.1 < \langle\omega_e\rangle/\langle\sigma\rangle < 10.$, the excess circulation of the merged daughter spot is at least 90% of the excess circulation of the parents. The resulting steady spot is surrounded by the chaotic flow of the detached filaments. We have found that if the initial impact parameter is larger than the characteristic size of the initial spots, the spots will not merge.

When large numbers (10-20) of positive spots are present initially, pairs with small impact parameters merge. Small perturbations of the initial locations can result in different numbers of spots in the final flow. For example, if the merging occurs so that there is never a large impact parameter among the spots, then all of the spots will merge into one. A small perturbation of the initial locations can cause the merging to occur so that all of the initial spots merge into two spots and these two spots have a large impact parameter and never merge together.

Merging of positive spots with $\langle\omega_e\rangle/\langle\sigma\rangle$ of order unity occurs on a turn-around timescale. If $\langle\omega_e\rangle/\langle\sigma\rangle \gg 1$ then the merging process, if it occurs at all, is qualitatively different. It is similar to the merging of vortices in the usual 2-dimensional vortex dynamics[5]; the timescales are slow and dominated by the small numerical dissipation present in the numerics.

2.4 One positive spot with varying ω_e

Experiments were performed with $C_2 = 0$ and with one initial spot of positive excess vorticity whose value we varied from $0.3\langle\sigma\rangle$ to $3.0\langle\sigma\rangle$. In particular we studied initially circular distributions of excess vorticity which decreased or increased linearly in radius of the spot. We found that in all cases the spot initially oscillated similar to a positive spot with a uniform ω_e. However the spots in these experiments always organized themselves so that the fluid elements with the larges value of ω_e were at the center with ω_e decreasing monotonically outward. Not surprisingly, the timescale for the vorticity redistribution was of order the spot turnaround time. This redistribution of ω_e is consistent with the merging experiments which showed that when fluid with $\omega_e = 0$ was entrained in a spot, it was quickly ejected.

2.5 Spots with $\beta = 0$ and $C_2 \neq 0$

All of the above experiments were repeated with $\beta = 0$ and $C_2 \neq 0$ corresponding to an annulus with a flat bottom but with a velocity $\tilde{u}$ with shear $-2C_2/r^2$. The shear has the same sign throughout the flow (but opposite in sign to C_2). When positive and negative excess vorticities are defined with respect to the sign of this shear, we find that all of the above numerically observed results are still true. This indicates that excess-vorticity dynamics does not depend on the cause of the shear in $\tilde{u}$ but only on its presence.

3. EQUILIBRIA OF SPOTS IN A SHEAR FLOW

If the size of a spot is small compared to the value of its radial location, then the variation of $\sigma(r)$ at the spot and the effects of curvature of $\tilde{u}$ are small. We can make a local Cartesian approximation to $\tilde{u}$ and find an analytic family of exact, linearly stable solutions to equations (1) and (2) consisting of a superposition of $\tilde{u}$ and an exactly elliptical spot of uniform excess vorticity with ellipticity (length of y-axis to x-axis) is[5,2]

$$\varepsilon = 2/\{1 + \sigma/\omega_e + [(1 + \sigma/\omega_e)^2 + 4\sigma/\omega_e]^{1/2}\} \qquad (16)$$

For $0.1 < \omega_e/\sigma < 10$. equation (16) is reasonably well approximated by our experimental relation (15). Equations (15) and (16) both show that in the absence of shear the spot is a circle, and as the shear is increased positive spots get increasingly stretched in the azimuthal direction. Equation (16) shows there are positive spots for all values of ω_e/σ but negative spots exist only for small values of $|\omega_e|$, in particular for $|\omega_e/\sigma| < 3 - 2\sqrt{2}$. Negative solutions can be shown to be unstable to small finite amplitude perturbations[2]. These instabilities can be understood physically by realizing that a negative spot is surrounded by a closed stream line or sepatrix with no fluid motion across it. All stream lines interior to the sepatrix are closed and have circulation with the same sign as ω_e; all stream lines exterior to the sepatrix are open, begin and end at infinity, and have circulation opposite in sign to ω_e. Any finite amplitude perturbation of an element of the spot that keeps it inside the sepatrix is bounded in the sense that the element remains near the spot for all time. An element perturbed outside the sepatrix gets advected to infinity and never returns to the spot (an instability). The sepatrix is never far from the boundary of the spot and therefore the negative spots are not stable to large amplitude perturbations.

4. MOMENTUM AND ENERGY CONSERVATION

To understand the stability and merging criteria of spots, we need to show how energy and momentum are conserved by equations (1)-(3). The angular momentum is equal to a constant plus a term proportional to

$$\xi = \int_{R_1}^{R_2} r dr \int d\phi \; \omega_e(r,\phi) r^2 \tag{17}$$

Thus, ξ which is equal to the excess-vorticity weighted value of r^2 is conserved in time. Energy is also conserved and is equal to a constant plus $E \equiv E_{self} + E_{int}$ where

$$E_{self} = -\frac{1}{8\pi} \int_{R_1}^{R_2} r dr \int d\phi \int_{R_1}^{R_2} r' dr' \int d\phi' \omega_e(\underline{r}) \omega_e(\underline{r}') \ln|\underline{r} - \underline{r}'| \tag{18}$$

$$E_{int} = \int_{R_1}^{R_2} r dr \int d\phi \, \tilde{\psi} \, \omega_e \tag{19}$$

and where

$$\tilde{u}(r) = \frac{d\tilde{\psi}(r)}{dr} \tag{20}$$

We have ignored the effects of the boundaries on E, but when ω_e is far from the boundaries this is a reasonable approximation[2]. Equations (18)-(19) show that there is an analogy between ω_e and an electric charge density in two dimensions and between $\tilde{\psi}$ and the potential of an applied electric field. With $\tilde{u} = 0$ or no applied electric field, equation (18) shows that energy is required from an external source to push together two spots of ω_e or charge of the same sign. Equation (19) shows that the sign of E_{int} depends on the sign of ω_e with respect to $\tilde{\psi}$. In particular, for flows discussed in this paper where either $\beta \neq 0$ and $C_2 = 0$ or $\beta = 0$ and $C_2 \neq 0$, the sign of E_{int} depends on the sign of $\langle\sigma\rangle\ \langle\omega_e\rangle$. A non-zero $\tilde{u}$ breaks the energy degeneracy between flows with positive and negative excess vorticity.

5. PREDICTION OF Γ_1, Γ_2, AND Γ_3 FOR NEGATIVE SPOTS

It seems plausible, given the initial values of Γ, ξ and E, to try to compute the three unknowns Γ_1, Γ_2, and Γ_3 that define the final time-averaged velocity $v(r)\hat{e}_\phi$ that forms after the break-up of negative spots. (See equation (13).) Assuming that v_f is ergodic, time averaging is the same as space averaging, so $\underline{v}_f$ by its definition has no excess circulation. Furthermore, if the filaments in $\underline{v}_f$ are macroscopically homogeneous, $\underline{v}_f$ cannot contribute to ξ. We therefore have two conservation equations useful for computing Γ_1, Γ_2, and Γ_3: equation (12) and

$$\xi = \Gamma_1 R_1^2 + \Gamma_2 R_2^2 + \Gamma_3 (R_2^2 + R_1^2)/2 \tag{21}$$

Ergoticity of $\underline{v}_f$ guarantees that the cross-product term in the kinetic energy integral between the time-averaged component of the velocity and $\underline{v}_f$ is equal to zero. Therefore we can write E as

$$E = E_{self}(\text{due to } v) + E_{self}(\text{due to } \underline{v}_f) + E_{int}(\text{due to } v) \tag{22}$$

where E_{self}(due to v) is computed from equation (18) by setting the excess vorticity that appears in the double integral equal to the component of ω_e due to v, and E_{self}(due to $\underline{v}_f$) is computed by using the component due to $\underline{v}_f$. Ergoticity also makes E_{int}(due to $\underline{v}_f$) equal to zero. Although E_{self}(due to v) and E_{int}(due to v) can be expressed in terms of Γ_1, Γ_2, and Γ_3, we have no *a priori* knowledge of the value of E_{self}(due to $\underline{v}_f$). The value of the latter energy is sensitive to the microscopic distribution of the filaments: decreasing the distance between pairs of filaments (while leaving the macroscopic density uniform) increases E_{self}(due to $\underline{v}_f$) logarithmically. Therefore, we cannot use energy conservation to help calculate Γ_1, Γ_2, and Γ_3.

We propose the hypothesis: The energy due to the time-averaged velocity

$[E_{self}(\text{due to } v) + E_{int}(\text{due to } v)]$ is minimized by the flow. Note that this hypothesis does *not* say that the energy in the large-scale component of the velocity is minimized. A spot with sharp edges has both large and small scales. We cannot derive our hypothesis from equations (1)-(3). Our motivation is loosely based on thermodynamics – energy is transferred irreversibly from ordered to disordered motion – with $\underline{v}_f$ identified as disorder. We now show the consequences of this hypothesis. We minimize the total energy in the large part of the flow

$$[E_{self}(\text{due to } v) + E_{int}(\text{due to } v)] =$$
$$[\Gamma_3^2(R_2^2+R_1^2)/(R_2^2-R_1^2) + 4\Gamma_1^2 \ln \frac{R_2}{R_1} + 4\Gamma_3\Gamma_1]/16\pi$$
$$+ \beta[\Gamma_3(R_2^5-R_1^5)/15(R_2^2-R_1^2) + \Gamma_1(R_2^3-R_1^3)/9]$$
$$+ C_2[\Gamma_3 + \Gamma_1 \ln \frac{R_2}{R_1}] \tag{23}$$

subject to the constraints of equations (12) and (21) and subject to the requirement that Γ_1, Γ_2 and Γ_3 must all be less than or equal to zero (due to the fact that $\omega_e \le 0.$). (Note that equation (23) is the exact energy and includes the effects of the boundaries.)

We have found that the minimization hypothesis when applied to the break up of a negative spot is correct. The values of Γ_1, Γ_2, and Γ_3 predicted by energy minimization agree within 5% (the numerical uncertainty) with the results of the numerical simulations of the initial value equations discussed in section 2.1.

6. ENERGY MINIMIZATION HYPOTHESIS APPLIED TO SPOT MERGERS

Note the following: If two ellipses of equal area, ellipticity, and radial location combine to form one ellipse with the same ellipticity, same total area and with radial location such that the merger conserves the second radial moment

$$\int r^2 \, d(\text{area}) \tag{24}$$

then the merger increases the cubic moment

$$\int r^3 \, d(\text{area}) \tag{25}$$

and decreases the logarithmic moment

$$\int \ln(r)\ d(\text{area}) \tag{26}$$

The integrals in equations (24)-(26) are evaluated over the area of the ellipse(s). Consider the merger of two initial spots of excess vorticity. For simplification of the analyses let both spots have the same excess vorticity ω_e uniform over the spot, let them have ellipticity given by equation (16). Let the spots have equal area, no impact parameter, and be sufficiently far from each other and the boundaries so that they are approximate equilibria. If the spots merge with no loss of ω_e into filaments, E_{self} increases. If $\beta \neq 0$ and $C_2 = 0$, then the increase in the cubic moment due to the merger (see equation (25)) causes E_{int}, as defined by equation (19), to decrease if ω_e has the same sign as σ (i.e., the same sign as β) and causes E_{int} to increase if ω_e has the opposite sign. If $\beta = 0$ and $C_2 \neq 0$, then the decrease in the logarithmic moment produces the same result: E_{int} decreases if ω_e has the same sign as σ. Therefore, two spots of negative excess vorticity always increase their energy by merging. In the absence of a $\underline{v}_f$ negative spots cannot merge and still conserve energy. In the presence of a $\underline{v}_f$ they can only merge if the spots (the time-averaged component of the velocity) can extract energy from $\underline{v}_f$. By our hypothesis, this is forbidden. In fact, because fragmentation of negative spots decreases the energy in the time-averaged component of the flow, the prediction of energy minimization is that all negative spots fragment. This is consistent with the numerical experiments.

The merger of two positive spots decreases E_{int} but increases E_{self}. From equations (18)-(19) we see that the ratio of E_{int} to E_{self} is $\langle\sigma\rangle/\langle\omega_e\rangle$ multiplied by a geometric factor. In a more detailed analysis this factor can be shown to decrease with increasing impact parameter[2]. Therefore if $\langle\sigma\rangle/\langle\omega_e\rangle$ is sufficiently large and the impact parameter is sufficiently small, E(due to v) can decrease. In the absence of $\underline{v}_f$ two positive spots can merge only if the change in E(due to v) is zero. The minimization hypothesis predicts that two positive spots will merge in the presence of a $\underline{v}_f$ if E(due to v) decreases. It is possible to derive semi-analytically the criterion for which two arbitrary initial positive spots will decrease E(due to v) by merging as a function of the initial impact parameter and $\langle\omega_e\rangle/\langle\sigma\rangle$. Within the numerical uncertainties of decomposing the velocity into a time-averaged component* and a $\underline{v}_f$, we find that the minimization hypothesis correctly predicts the circumstances under which positive spots merge in our initial value experiments.

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*Time-averaging is done on time scales of 1 to 3 turnaround times.

ACKNOWLEDGMENT

We wish to thank the National Center for Atmospheric Research (operated by NSF) for computation support. We acknowledge NSF grants AST851512, MSM8515289, ONR grant N00014-86-M-0385 and Lawrence Livermore National Laboratory subcontract 7802105.

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