Selection Rules for the Nonlinear Interaction of Internal Gravity Waves

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Two intersecting beams of internal gravity waves will generically create two wave packets by nonlinear interaction. The frequency of one packet will be the sum and that of the other packet will be the difference of the frequencies of the intersecting beams. In principle, each packet should form an ''X'' pattern, or ''St. Andrew's cross'' consisting of four beams outgoing from the point of intersection. Here we derive selection rules and show that most of the expected nonlinear beams are forbidden. These rules can also be applied to the reflection of a beam from a boundary.

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Introduction.—Interacting internal gravity waves in stratified fluids have been studied for over a hundred years [1]. Thus, it is surprising that there is a set of previously unknown selection rules that forbids most of their nonlinear interactions. Internal gravity waves are currently of interest because their reflections (i.e., interactions with their ''images'') from the ocean floor may sculpt continental shelves [2]. Also, our numerical experiments have shown that interacting internal gravity waves can form vortices in protoplanetary disks [3].

A two-dimensional, compact source of gravity waves oscillating at frequency $\pm\omega$ creates four, columnated outgoing beams in an ''X'' pattern, as in Fig. 1, known as a ''St. Andrew's cross'' [4]. There are four beams, or ''legs'', because the angle θ of each beam with respect to the positive x axis obeys the dispersion relation

$$|\omega|/N = |\sin\theta|, \tag{1}$$

where $N \equiv \sqrt{-g(d\bar{\rho}/dz)/\rho_0}$ is the Brunt-Väisälä frequency, $d\bar{\rho}/dz$ is the vertical (z) density gradient of the unperturbed fluid, g is the acceleration of gravity, and ρ_0 is the average density. Intersecting beams can produce first harmonics with frequencies ω equal to the sum or difference of the frequencies of the interacting beams, subject to the *solvability condition* imposed by Eq. (1): $0 < |\omega|/N \leq 1$. Thus, two interacting beams should produce a St. Andrew's cross with a ''low frequency'' equal to the difference of the absolute values of the frequencies of the interacting beams. A second St. Andrew's cross with a ''high frequency'' equal to the sum of the absolute values of the frequencies of the interacting beams will also be created if the solvability condition is satisfied. Thus, one expects either four or eight harmonic beams or legs. However, in many simulations and experiments of interacting beams of internal gravity waves (e.g., Figs. 2(a), 2(b) in [5]; Fig. 2(b) in [6]; and Fig. 3(a) in [7]), one or more legs are missing. Fig. 2 shows another example of missing beams: the two primary beams have the same frequencies, $\pm\omega$, so the low frequency St. Andrew's cross cannot form. However, we would expect the beams to pass through each other, interact and create a high frequency cross because $|\omega| < N/2$. In Fig. 2(a) the sources of the primary beams are located at the right-side corners, and only two of the four expected legs are created. In Fig. 2(b) the sources of the primary beams are at the top, and *no* legs are created. Tabaei *et al.* [8] found selection rules governing the creation of harmonic legs that correctly predicted that the interaction in Fig. 2(a) creates only two, of the possible four, legs. However, their rules are incomplete; they also predicted that the interaction in Fig. 2(b) would create all four legs [Fig. 6(b) and Table I in [8]].

Here we derive the complete set of selection rules for interacting beams of gravity waves. A beam consists of a packet of complex conjugate pairs of plane waves $e^{i(k_x x + k_z z + \omega t)}$ with a continuum of wave numbers constrained such that all of the waves' group velocity vectors $\mathbf{c}$ point in the same direction. From Eq. (1) the absolute value of the frequencies of all plane waves in a beam must

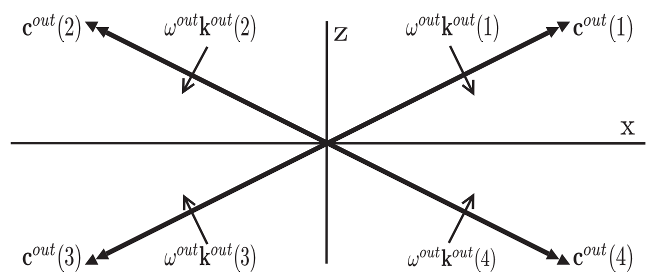


FIG. 1. St. Andrew's cross of beams propagating from the origin. Each beam is labeled with its quadrant n. Each thick, double-headed arrow represents a group velocity $\mathbf{c}^{\rm out}(n)$ of a typical plane wave in the beam. Each thin, single-headed arrow shows the product of that plane wave's wave vector $\mathbf{k}^{\rm out}(n)$ and frequency $\omega^{\rm out}$. Although a beam is a continuum of plane waves, the group velocity vectors of all of a beam's plane waves point in the same direction $\theta^{\rm out}(n)$. In this ''thought experiment'', all plane waves in a beam have the same frequency $\omega^{\rm out}$; i.e., the beams are not made of complex conjugate pairs of waves. Relative directions of the thin and thick vectors are drawn consistent with Eqs. (2) and (3).

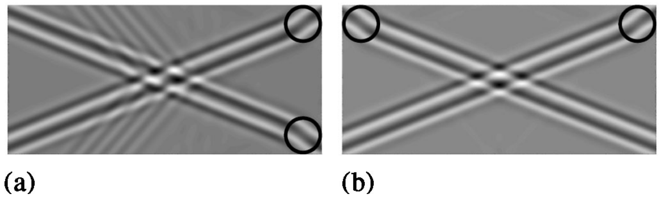


FIG. 2. Numerical simulations of physical beams, shown by the magnitude of their vertical velocities. Each primary beam has frequencies of $\pm\omega$ with $|\omega|/N = 0.3746 < 1/2$. (a) (left) The primary beam sources lie within the circles in the corners on the right side of the panel. Both primary beams propagate to the left, interact, and create two harmonic beams or legs with frequencies $\pm 2\omega$. (b) (right) As in panel (a), but with sources at the top. No harmonic beams are produced.

be the same. It is important to understand the relative directions of the group velocity vector $\mathbf{c}$ and wave vector $\mathbf{k}$. For internal gravity waves:

$$\mathrm{sgn}\{c_x\} = \mathrm{sgn}\{\omega k_x\} \tag{2}$$

$$\mathrm{sgn}\{c_z\} = -\mathrm{sgn}\{\omega k_z\}. \tag{3}$$

To prove Eqs. (2) and (3), note that the dispersion relation written in its traditional form is

$$\omega = \pm N k_x / k. \tag{4}$$

This dispersion relation and the definition $\mathbf{c} \equiv \nabla_{\mathbf{k}} \omega$ show

$$\omega k_x = c_x (k k_x / k_z)^2 \qquad \omega k_z = -c_z k^2, \tag{5}$$

which proves $(\mathbf{k} \cdot \mathbf{c}) = 0$ and relations (2) and (3). The definition of θ and Eq. (5) show that for all beams:

$$\cot\theta \equiv c_x / c_z = -k_z / k_x. \tag{6}$$

Note that Eq. (1) follows from Eqs. (4) and (6). Equations (4) and (5) show that we must exclude the case when $\omega = 0$ (i.e., $\theta = 0$, or π) because no plane waves exist and also the case when $|\omega| = N$ (i.e., $\theta = \pm\pi/2$) because $\mathbf{c} = 0$, and we are only interested in propagating beams.

Although a beam consists of complex conjugate pairs of waves, consider a thought experiment in which each beam has only a positive *or* a negative frequency. Because the linearized equations for gravity waves are reflection symmetric about the x and z axes, we let one beam, labeled as the zeroth beam, approach the origin from the first quadrant. We use the notation that it has frequency $\omega^{\mathrm{in}}(0)$, wave vector $\mathbf{k}^{\mathrm{in}}(0)$, angle $\theta^{\mathrm{in}}(0)$, and group velocity $\mathbf{c}^{\mathrm{in}}(0)$ pointing toward the origin. At the origin this beam intersects a second beam, also pointing toward the origin, with frequency $\omega^{\mathrm{in}}(j)$, wave vector $\mathbf{k}^{\mathrm{in}}(j)$, group velocity $\mathbf{c}^{\mathrm{in}}(j)$, and angle $\theta^{\mathrm{in}}(j)$ with $j = 1, 2, 3$, or 4. We use the notation that the second beam lies in the jth quadrant. The inviscid, linearized equations for gravity waves are reversible in time, so we can require $\omega^{\mathrm{in}}(0)$ to be positive, and by using the reflection symmetries we can also require that $\omega^{\mathrm{in}}(0) \geq |\omega^{\mathrm{in}}(j)|$ for $j = 1, \ldots, 4$. Similarly, the four harmonic outgoing beams generated at the origin (Fig. 1) have frequency ω^{out}, wave vector $\mathbf{k}^{\mathrm{out}}(n)$, group velocity $\mathbf{c}^{\mathrm{out}}(n)$, and angle $\theta^{\mathrm{out}}(n)$, where n is the quadrant that the beam propagates into, and $n = 1, \ldots, 4$. Because the outgoing waves in the nth beam are generated from the waves in the incoming zeroth and jth beams by quadratic nonlinearities:

$$\omega^{\mathrm{out}} = \omega^{\mathrm{in}}(0) + \omega^{\mathrm{in}}(j) \tag{7}$$

$$\mathbf{k}^{\mathrm{out}}(n) = \mathbf{k}^{\mathrm{in}}(0) + \mathbf{k}^{\mathrm{in}}(j). \tag{8}$$

Note that ω^{out} is independent of the quadrant n of the outgoing beam. Using Eqs. (1) and (7), we obtain

$$\cot[\theta^{\mathrm{out}}(n)] = -(-1)^n \sqrt{N^2/[\omega^{\mathrm{in}}(0) + \omega^{\mathrm{in}}(j)]^2 - 1}. \tag{9}$$

The sign of the right-hand side of Eq. (9) is obtained by satisfying the geometric tautology that $\mathrm{sgn}\{\cot[\theta^{\mathrm{out}}(n)]\} = -(-1)^n$. From Eqs. (6) and (8):

$$k_z^{\mathrm{in}}(0) + k_z^{\mathrm{in}}(j) = -\cot[\theta^{\mathrm{out}}(n)][k_x^{\mathrm{in}}(0) + k_x^{\mathrm{in}}(j)]. \tag{10}$$

Using Eq. (6) to express the x components of $\mathbf{k}^{\mathrm{in}}(0)$ and $\mathbf{k}^{\mathrm{in}}(j)$ in terms of their z components, we multiply both sides of Eq. (10) by $\omega^{\mathrm{in}}(0)$ to obtain:

$$[\omega^{\mathrm{in}}(0) k_z^{\mathrm{in}}(0)]\beta(n, j) = -[\omega^{\mathrm{in}}(0)/\omega^{\mathrm{in}}(j)][\omega^{\mathrm{in}}(j) k_z^{\mathrm{in}}(j)], \tag{11}$$

where

$$\beta(n, j) \equiv \frac{1 - \cot[\theta^{\mathrm{out}}(n)]\tan[\theta^{\mathrm{in}}(0)]}{1 - \cot[\theta^{\mathrm{out}}(n)]\tan[\theta^{\mathrm{in}}(j)]}. \tag{12}$$

First selection rule.—We obtain our first selection rule by requiring that the signs of both sides of Eq. (11) are the same. Equation (3) shows that $\mathrm{sgn}\{\omega^{\mathrm{in}}(0) k_z^{\mathrm{in}}(0)\} = -\mathrm{sgn}\{c_z^{\mathrm{in}}(0)\}$. Because the zeroth incoming beam lies in the 1st quadrant, $\mathrm{sgn}\{c_z^{\mathrm{in}}(0)\} = -1$. Thus, Eq. (11) becomes

$$\mathrm{sgn}\{\beta(n, j)\} = \mathrm{sgn}\{\chi(j)\}\mathrm{sgn}\{c_z^{\mathrm{in}}(j)\}, \tag{13}$$

where $\chi(j) \equiv \omega^{\mathrm{in}}(j)/\omega^{\mathrm{in}}(0)$. Note that $\mathrm{sgn}\{c_z^{\mathrm{in}}(j)\}$ is negative for $j = 1, 2$ and positive for $j = 3, 4$. The sign of β is found on a case by case basis:

Beams with $\mathrm{sgn}\{\chi(j)\} = +1$.—For example, when $\mathrm{sgn}\{\chi(j)\} = +1$, we can show that $\mathrm{sgn}\{\beta(n, j)\} = +1$, for all n and j. To see this, define a function $f(\gamma) \equiv \sqrt{1/\gamma - 1}$. Note that $df/d\gamma < 0$, and that

$$\begin{aligned} f(\gamma_1) &= |\cot[\theta^{\mathrm{in}}(0)]| \quad \text{for } \gamma_1 \equiv [\omega^{\mathrm{in}}(0)/N]^2 \\ f(\gamma_2) &= |\cot[\theta^{\mathrm{in}}(j)]| \quad \text{for } \gamma_2 \equiv [\omega^{\mathrm{in}}(j)/N]^2 \\ f(\gamma_3) &= |\cot[\theta^{\mathrm{out}}(n)]| \\ &\quad \text{for } \gamma_3 \equiv [\omega^{\mathrm{in}}(0)/N]^2 + [\omega^{\mathrm{in}}(j)/N]^2 \\ &\qquad + 2\mathrm{sgn}\{\chi(j)\}|\omega^{\mathrm{in}}(0)/N||\omega^{\mathrm{in}}(j)/N|. \end{aligned} \tag{14}$$

Because $\gamma_1 < \gamma_3$ and because $df/d\gamma < 0$, we can show that $|\cot[\theta^{\mathrm{out}}(n)]\tan[\theta^{\mathrm{in}}(0)]| < 1$. Thus, the numerator of

$\beta(n, j)$ in Eq. (12) is positive. Similarly, because $\gamma_2 < \gamma_3$, the denominator of β is also positive. This shows that $\mathrm{sgn}\{\beta\} = +1$. Therefore, the signs in Eq. (13) are consistent if and only if $c_z^{\mathrm{in}}(j) > 0$, or equivalently, if and only if $j = 3, 4$. Thus, in this thought experiment, harmonic beams can be created only when $j = 3$ or $j = 4$, but this rule imposes no restriction on the quadrant n of the beam produced by the interaction. Note this rule (i.e., that $j = 3$ or $j = 4$) explains why the interaction in Fig. 2(b) creates no harmonic beams.

Beams with $\mathrm{sgn}\{\chi(j)\} = -1$.—In this case, we have $\gamma_3 > \gamma_1$ if and only if $|\omega^{\mathrm{in}}(j)| > 2|\omega^{\mathrm{in}}(0)|$. However, this is never true because we assumed that $|\omega^{\mathrm{in}}(j)| \leq |\omega^{\mathrm{in}}(0)|$, for $j = 1, \ldots, 4$. Thus, the sign of the numerator of β is $(-1)^n$ for all j and n.

Beams with $\mathrm{sgn}\{\chi(j)\} = -1$ *and* $|\omega^{\mathrm{in}}(j)| < |\omega^{\mathrm{in}}(0)|/2$.—In this case, $\gamma_3 > \gamma_2$. Thus, the denominator of β is positive for all n and j, and $\mathrm{sgn}[\beta(n, j)] = (-1)^n$. Therefore, Eq. (13) shows: for $j = 1$ or $j = 2$, n must be even; and for $j = 3$ or $j = 4$, n must be odd.

Beams with $\mathrm{sgn}\{\chi(j)\} = -1$ *and* $|\omega^{\mathrm{in}}(j)| > |\omega^{\mathrm{in}}(0)|/2$.—Here, the sign of the denominator of $\beta(n, j)$ is $(-1)^{1+n+j}$, so $\mathrm{sgn}[\beta(n, j)] = (-1)^{1+j}$ for all n. Equation (13) shows that beams can be produced in all quadrants if $j = 1$, or $j = 4$, and never produced if $j = 2$ or $j = 3$.

Beams with $\mathrm{sgn}\{\chi(j)\} = -1$ *and* $|\omega^{\mathrm{in}}(j)| = |\omega^{\mathrm{in}}(0)|/2$.—In this case, when $n + j$ is even, no harmonic beams are created. This is due to the fact that in this case the denominator in β in Eq. (12) is zero, and Eq. (11) shows that $k_z^{\mathrm{in}}(0) = 0$, so, $\mathbf{c}^{\mathrm{in}}(0) = 0$ which corresponds to a nonpropagating beam and is not a case of interest. When $n + j$ is odd, then $\mathrm{sgn}\{\beta(n, j)\} = (-1)^n$. Equation (13) is satisfied only when $j = 1$ and n is even, or it is satisfied when $j = 4$ and n is odd. Thus, no beams are ever generated if $j = 2$ or $j = 3$.

Although our analysis used positive $\omega^{\mathrm{in}}(0)$, our results extend to negative values. It can be shown that the selection rules depend only on the *relative* signs between $\omega^{\mathrm{in}}(0)$ and $\omega^{\mathrm{in}}(j)$, for $j = 1, \ldots, 4$, i.e., on $\mathrm{sgn}\{\chi(j)\}$.

Second selection rule.—A second rule comes from requiring that the signs on both sides of Eq. (8) are the same. Both the x and z components must be satisfied. Again, this selection rule must be examined on a case by case basis. However, for half the cases, the two terms on the right-hand side of Eq. (8) have the same sign, and the rule can be determined by "inspection". For example, consider the z component of Eq. (8) for an incoming beam with $j = 4$ and with $\mathrm{sgn}\{\omega^{\mathrm{in}}(j)\} = -1$. Because $\mathrm{sgn}\{c_z^{\mathrm{in}}(0)\} = -1$ and because $\mathrm{sgn}\{\omega^{\mathrm{in}}(0)\} = +1$, Eq. (3) shows that $\mathrm{sgn}\{k_z^{\mathrm{in}}(0)\} = +1$. Equation (3) also shows that $\mathrm{sgn}\{k_z^{\mathrm{in}}(4)\} = +1$. Therefore, Eq. (8) shows that $\mathrm{sgn}\{k_z^{\mathrm{out}}(n)\} = +1$. By assumption, $|\omega^{\mathrm{in}}(0)| \geq |\omega^{\mathrm{in}}(4)|$, so Eq. (7) shows that $\mathrm{sgn}\{\omega^{\mathrm{out}}\} = +1$. Therefore $\mathrm{sgn}\{c_z^{\mathrm{out}}(n)\} = -1$, and n must be 3 or 4.

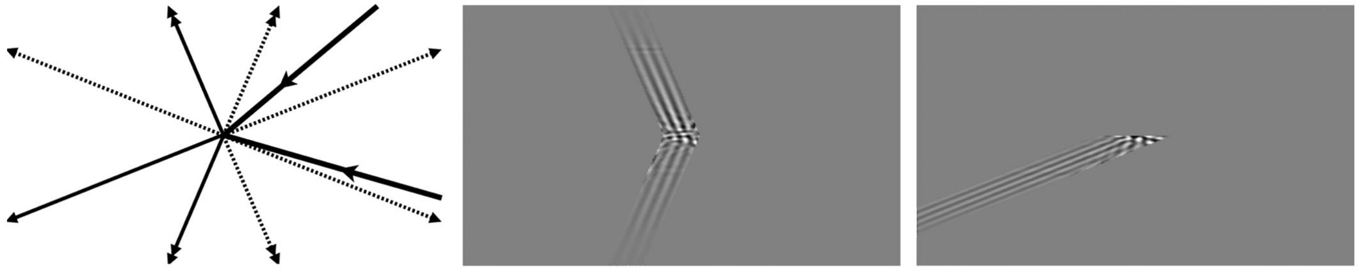
For cases in which the two terms on the right-hand side of Eq. (8) have opposite signs, the sign of the right-hand side is not obvious, and the 2nd Selection Rule cannot be determined by inspection. However, even in this case, the sign of the right-hand side of Eq. (8) can be determined and the 2nd Selection Rule found. The trick is to note that the sign of the right-hand side depends on whether $|k_z^{\mathrm{in}}(j)/k_z^{\mathrm{in}}(0)| \equiv |\beta(n, j)|$ and $|k_x^{\mathrm{in}}(j)/k_x^{\mathrm{in}}(0)| \equiv |\beta(n, j) \times \tan[\theta^{\mathrm{in}}(j)]/\tan[\theta^{\mathrm{in}}(0)]|$ are greater than or less than unity. Finding these values is easy, but must be done on a case by case basis similar to the way we determined $\mathrm{sgn}\{\beta(n, j)\}$ above. The results of all of our selection rules are summarized in Table I.

Application of selection rules to physical beams.—A physical beam of gravity waves, rather than a thought experiment, consists of complex conjugate pairs of waves. To exist, the high frequency St. Andrew's cross requires $[|\omega^{\mathrm{in}}(0)| + |\omega^{\mathrm{in}}(j)|] < N$. The rules for that cross are in the first row of Table I: the high frequency cross can only be produced if $j = 3$ or 4, and has only two legs (with $n = 2$, 3). Using reflection symmetries, we generalize these statements so that no beam needs to be in the first quadrant: a high frequency cross is produced only if one of the incoming beams propagates upward and the other downward. The cross has only two outgoing beams—one propagates upward and the other downward. Both outgoing beams propagate horizontally in the same direction as the incoming beam with the higher absolute value of its frequency. If the two incoming beams have the same frequency and propagate in opposite directions, then no high frequency beams, or legs, are produced.

A low frequency St. Andrew's cross has two, or fewer legs and cannot form if $|\omega^{\mathrm{in}}(0)| = |\omega^{\mathrm{in}}(j)|$. The rules for that cross are in the bottom three rows of Table I. As above, we use reflection symmetries to generalize the rules. The second row of Table I is the case when the absolute value of the frequency of the incoming beam with lower frequency is less than half the absolute value of the frequency of the beam with higher frequency. In this case, a low frequency

TABLE I. Selection rules for creating harmonic beams from two primary beams intersecting at the origin. One incoming beam, labeled as the zeroth lies in the first quadrant with frequency $\omega^{\mathrm{in}}(0)$. The second incoming beam lies in the jth quadrant with frequency $\omega^{\mathrm{in}}(j)$. Depending on the value of $\chi \equiv \omega^{\mathrm{in}}(j)/\omega^{\mathrm{in}}(0)$, there are four possible scenarios, indicated by each of the four rows of the table. The first two columns specify χ. (Without loss of generality, $|\chi| \leq 1$.) For each row, the quadrant numbers n of the allowable outgoing beams are listed as a function of j (column). *Solvability* requires $|\omega^{\mathrm{in}}(j) + \omega^{\mathrm{in}}(0)| < N$. This table uses both the first and second selection rules. For a harmonic beam to exist, it must satisfy both rules. When $\mathrm{sgn}\{\chi\} = +1$ and $j = 3$, no harmonic beams are produced if $\omega^{\mathrm{in}}(3) = \omega^{\mathrm{in}}(0)$.

$\mathrm{sgn}\{\chi\}$	range of $\lvert\chi\rvert$	$j = 1$	$j = 2$	$j = 3$	$j = 4$
+1		none	none	2, 3	2, 3
−1	$\lvert\chi\rvert < 1/2$	2	2	3	3
−1	$\lvert\chi\rvert = 1/2$	2	none	none	3
−1	$1/2 < \lvert\chi\rvert \leq 1$	1, 2	none	none	3, 4



(a) (b) (c)

FIG. 3. Schematic and a numerical simulation of physical beams verifying the selection rules in the fourth column and top two rows of Table I. Here, $\theta^{\rm in}(0) = 40°$ and $\theta^{\rm in}(4) = -16°$, so $|\chi| < 1/2$. (a) (left) Schematic shows: two incoming primary beams from the right as heavy arrows; the three allowable (according to Table I) outgoing harmonics as thin solid arrows; and the five disallowed harmonics as dotted arrows. The high frequency harmonic St. Andrew's cross has two arrowheads and the low frequency harmonic has one. (b) (center) The vorticity in a numerical simulation with output filtered to show only frequencies with $\omega^{\rm out} = \pm(|\omega^{\rm in}(0)| + |\omega^{\rm in}(4)|)$, i.e., the high frequency harmonic or top row of Table I. (c) (right) As in (b), but filtered to show frequencies $\omega^{\rm out} = \pm(|\omega^{\rm in}(0)| - |\omega^{\rm in}(4)|)$, i.e., the low frequency harmonic or second row of Table I.

cross with only one leg is always generated. The outgoing beam propagates in the same horizontal direction as the incoming beam with higher frequency. The outgoing beam propagates vertically in the direction opposite that of the vertical direction of the incoming beam with lower frequency.

The third row of Table I is for a low frequency cross created when the absolute value of the frequency of the incoming beam with lower frequency is equal to half the absolute value of the frequency of the beam with higher frequency. Here, a low frequency cross is created only when both incoming beams propagate in the same horizontal direction. The cross has only one beam. It propagates horizontally in the direction of the incoming beams. The outgoing beam propagates vertically in a direction opposite that of the incoming beam with lower frequency.

The bottom row of Table I is for the case when the absolute value of the frequency of the incoming beam with lower frequency is greater than half the absolute value of the frequency of the beam with higher frequency. In this case, a low frequency cross is generated only when the two incoming beams propagate in the same horizontal direction. The cross has two legs. They propagate vertically in the direction opposite that of the incoming beam with lower frequency. The two outgoing legs propagate horizontally in opposite directions.

Discussion.—All of our selection rules have been verified numerically (e.g., Fig. 3). Our rules are *complete* because no harmonic beams allowed by Table I are missing from our numerical simulations. (We tested all possibilities). The selection rule in [8] is a subset of our second selection rule; however, the rule in [8] requires only that the signs on both sides of the x component (but not z component) of Eq. (8) are the same. Moreover, in [8] the rule is stated only for the simple case in which signs can be determined by inspection. Because the reflection of a beam from a boundary can be computed as the interaction of a beam with its ''image'', our selection rules can be applied to reflection, and those rules will be presented in a future paper.

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