Distinct Quasiperiodic Modes with Like Symmetry in a Rotating Fluid

K. T. Coughlin and P. S. Marcus
Department of Mechanical Engineering, University of California, Berkeley, California 94720

R. P. Tagg[(a)] and Harry L. Swinney
Center for Nonlinear Dynamics and the Department of Physics, University of Texas, Austin, Texas 78712
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We use Floquet theory, fully resolved numerical simulation, and laboratory experiment to study two quasiperiodic modes in the Couette-Taylor system which have the same symmetry but different spatial structure. Although each mode has been observed in previous experiments, their coexistence in the same region of parameter space is a new numerical prediction. Because they have the same symmetry, the modes are strongly coupled, which results in chaotic behavior.

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The study of temporally quasiperiodic fluid flows has focused primarily on their space-time symmetries, with less attention paid to the physical characteristics of the unstable modes. In systems with circular symmetry, such as Couette-Taylor flow in which fluid is confined between concentric rotating cylinders, generically[1] one finds bifurcations from temporally periodic rotating waves to quasiperiodic modulated waves. Mathematically, the transition is characterized by the change in the symmetry group that leaves the flow invariant.[2] The question of whether there may be physically different modes that have the same symmetry has not been previously addressed. The existence of qualitatively different modes with the same space-time symmetry is common in fluid dynamics,[3] but is often overlooked in hydrodynamic stability theory. This multiplicity can have important consequences for the dynamics, and possibly the transition to chaos, in this system.

In this paper, we demonstrate the existence of two distinct quasiperiodic Couette-Taylor modes, describe their structure, and discuss the implications of their coexistence over a certain parameter range. We begin by using Floquet theory to derive the unique functional form for quasiperiodic solutions to the Navier-Stokes equations. We then present results from numerical simulations illustrating the two quasiperiodic flows and results from the experiments corroborating the numerical observations. We end with a discussion of how these modes may be used to derive from the Navier-Stokes equation a low-dimensional spectral truncation that can be quantitatively compared with full numerical simulations.

Linearization of the Navier-Stokes equations around the rotating-wave state gives a Floquet system, linear equations with coefficients that are periodic in $\phi - c_1 t$, where c_1 is the angular phase speed of the wave. (We nondimensionalize by setting the inner cylinder speed, the gap width, and the density equal to 1.) If we assume that the bifurcation to quasiperiodic flow preserves the axial periodicity, and that axially traveling modes do not occur, the linear Floquet modes will have the form $e^{im_2(\phi - c_2 t)}\mathbf{f}(r, \phi - c_1 t, z)$, where c_2 is real at onset. The function $\mathbf{f}$ is defined to have the symmetry of the rotating wave; hence, it can be written as

$$\mathbf{f} = \sum_j \mathbf{b}_j(r,z) e^{ijm_1(\phi - c_1 t)}, \tag{1}$$

with m_1 equal to the number of waves in the base flow. The functional form of the full solution for the velocity past onset must be invariant with respect to the equations of motion; hence, it will consist of a sum over the set of spectral functions closed under the action of the nonlinear terms. It can be written as a double Fourier sum

$$\mathbf{v} = \sum_{j,k} \mathbf{a}_{jk}(r,z) e^{ijm_1(\phi - c_1 t)} e^{ikm_2(\phi - c_2 t)}. \tag{2}$$

The parameter m_2 is not uniquely defined by the form of the solution, since any $m_2' = m_2 + pm_1$ for integer p in (2) reproduces the same functional form. This is consistent with the work of Rand,[1] which describes the symmetry classes for fixed values of m_1 and $m_2 \pmod{m_1}$. There is, however, a unique and physically meaningful definition of m_2 (and consequently of c_2), as discussed below. Mathematically, m_2 is defined such that the coefficients $\mathbf{a}_{jk}$ decay monotonically with $|j|$ and $|k|$. Despite the symmetry of (2) under $1 \leftrightarrow 2$, m_1 and m_2 are not physically equivalent. Even with the parameters m_1, c_1, m_2, and c_2 uniquely determined, the solution in (2) leaves the radial-axial structure of the flow completely unspecified (other than the requirement that it be axially periodic). This allows for the existence of a variety of physically different Floquet modes, which will be distinguished not by symmetry but by the characteristic values of the phase speeds and spatial structure of the modulation.

We have numerically solved the Navier-Stokes equations in an axially periodic, cylindrical geometry using a fully resolved (32^3 modes) pseudospectral method.[4,5] We hold the outer cylinder fixed and vary the Reynolds number $R \equiv \Omega a(b-a)/\nu$, where Ω is the inner cylinder frequency, a and b are the inner and outer cylinder radii, and ν is the kinematic viscosity. We impose an axial wavelength λ of 2.5 times the gap width, and fourfold azimuthal symmetry; hence, for our solutions m_2 is a multiple of $m_1=4$

The basic structure of the modulated flow is set by the bifurcation (at $R \equiv R_c$) to Taylor vortices, a three-dimensional, axisymmetric flow consisting of toroidal vortices wrapped around the inner cylinder. The vortices are separated by radial inflow and outflow boundaries, which advect the fast (slow) moving fluid near the inner (outer) cylinder, producing strong azimuthal jets with large axial gradients. A rotating wave, traveling at speed c_1, is visible as an approximately sinusoidal distortion of these boundaries. The modulation is identified by defining the quasiperiodic disturbance field as the total minus the time-averaged flow in the frame rotating with the underlying wave; at onset this is equal to the Floquet mode. Numerically, we find that the disturbance is always concentrated in the vicinity of the outflow boundary. The typical structure, as shown in Fig. 1, is a set of compact three-dimensional vortex pairs of alternating sign. We identify m_2 as the number of vortex pairs per wavelength $2\pi/m_1$ of the wave. These drift, with some distortion, along the outflow boundary contour at the mean speed c_2. Different quasiperiodic modes are distinguished by the ratio of phase speeds c_2/c_1, the detailed spatial structure of the Floquet mode, and the characteristic amplitude of modulation. These result in distinct visual signatures in the laboratory.

With increasing R we find three solution regimes. At $R/R_c=8.50$, we find an example of a mode first observed by Zhang and Swinney[6] (ZS) for which $c_2/c_1 \simeq 2$. The disturbance vorticity is weak, relatively fast moving, and spatially compact. Zhang and Swinney described this mode as a nontraveling modulation[6] because the strongest peak in the power spectrum, other than m_1c_1 and its harmonics, was the frame-independent beat frequency between m_2c_2 and m_1c_1 (see below and Fig. 3).

At a Reynolds number of $R/R_c=9.80$ we numerically find an example of a GS mode as first described by Gorman and Swinney,[7] for which $c_2/c_1 \simeq 4/3$. These modes have also been described as two-traveling-wave flows.[8] However, from (2) we realize that there exist a countably infinite number of rotating frames in which the flow is temporally periodic[4] [for example, in the frame rotating at speed c_1 the flow is periodic with a frequency $m_2(c_2-c_1)$], but there is no frame in which the flow is a traveling wave. In GS flow, the quasiperiodic vorticity also consists of concentrated vortex pairs located at the outflow, with characteristically larger modulation amplitude. Unlike the ZS flow, the wavy outflow boundaries of the GS flow with $m_1=m_2=4$ are observed to flatten periodically.

At $R/R_c=8.50$ the GS mode is visible as a decaying transient, implying that the growth rate of the Floquet mode is negative. Similarly, at $R/R_c=9.80$, the transients are dominated by the decaying ZS mode. Numerically, we find that there is an intermediate range of R near $9R_c$ where both modes are present with positive growth rate. In the frame rotating with the original wave, the flow is dominanted by two incommensurate frequencies; in the inertial frame, by three. Time series illustrating these three regimes are shown in Fig. 2. The

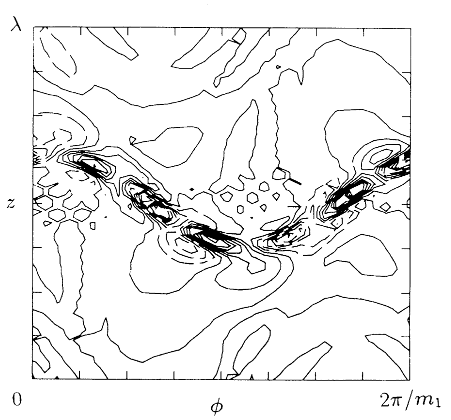

FIG. 1. A representation of the ZS Floquet mode, with the azimuthal vorticity projected onto an azimuthal-axial plane at midgap. Dashed contours represent negative values, solid contours non-negative. The contour interval is 0.04. The outflow jet is near $z=\lambda/2$. For this flow $m_2=3m_1$; there are three vortex pairs per wavelength $2\pi/m_1$ of the wave.

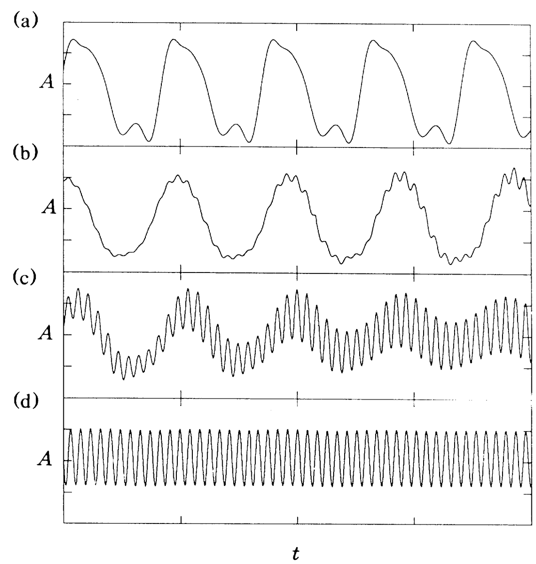

FIG. 2. Time series of the modulus A of the fundamental (in both the axial and azimthal directions) spatial Fourier mode at a fixed radial point. The length of the time series shown is approximately 11 inner-cylinder periods. (a) The equilibrium GS model at $R/R_c=9.80$. (b) The mixed state at $R/R_c=9.00$ with both ZS and GS components. (c) The flow at $R/R_c=8.70$, showing the decaying GS transient. (d) The flow in (c) after convergence to a ZS equilibrium.

long-term temporal behavior of the flow over the range in which the ZS and GS modes coexist is as yet undetermined. To generate the short-time series in Fig. 2 using full simulation requires 1.5 Cray-YMP computer hours; hundreds of Cray hours would be required to produce a time series long enough for the standard diagnostics of chaotic dynamics to be useful. This is not an efficient use of computer resources, nor does it make use of spatial information about the flow to illuminate the physical processes involved. These issues are discussed further below.

Our numerical work constitutes the first observation of a ZS mode in a simulation of the Navier-Stokes equations, and the first identification of the spatial character of the modulation. Experimental work has subsequently confirmed these simulations. The principal experimental results are that both the ZS and GS modes are robust and clearly distinguished by their spatial character and power spectra, that there is a transition between them, and that there is a range of R in which the flow is aperiodic over long time scales with the spectra containing features of both modes.

The experiments were conducted in a system of radius ratio 0.876, with the height of the fluid equal to 40 times the gap width of 0.740 cm. The ends of the annulus were solid rings fixed to the outer cylinder. The working fluid was water containing a 2% mixture of AQ1000 Rheoscopic Concentrate[9] for flow visualization. The initialization procedure was chosen[10] so as to obtain an axial wavelength of $\lambda=2.51$. At $R/R_c=9.80$, the experimental GS mode with $m_1=m_2=4$ has $c_1=0.334$ and $c_2/c_1=1.27$, compared with the numerically computed values of 0.331 and 1.31, respectively. The largest peak in the GS spectrum in Fig. 3(a), other than m_1c_1 and its harmonics, is the m_2c_2 peak. (Under the conditions of this experiment, a pure GS mode has not been obtained for times longer than a few hundred inner-cylinder periods.)

A pure ZS flow has been observed experimentally at the numerically predicted value of $R/R_c=8.50$ with $m_1=4$ and $m_2=12$. The experimental values of c_1 and c_2/c_1 are 0.336 and 2.1, compared with the numerical values of 0.338 and 2.02. Figure 3(c) shows that the largest spectral peak (other than m_1c_1 and its harmonics) is the rotating-frame-independent beat frequency $m_2(c_2-c_1)$. This peak is used to determine the value of m_2. The ZS mode is characterized by small-scale structure on the outflow boundary, which is absent from GS modes. In the experiment, there is a hysteretic transition between the ZS and GS modes, with a transition from a ZS to a GS flow at $R/R_c=9.76$, and a transition from a GS to a ZS flow at $R/R_c=9.58$. At $R/R_c=9.00$, the flow is temporally intermittent. The power spectrum in Fig. 3(b) indicates that both the m_2c_2 GS peak and ZS beat peak are present, consistent with the numerical results.[11] From the experimental spectra, $c_1=0.335$, and $c_2/c_1=1.3$ for the GS mode and 2.0 for the ZS mode.

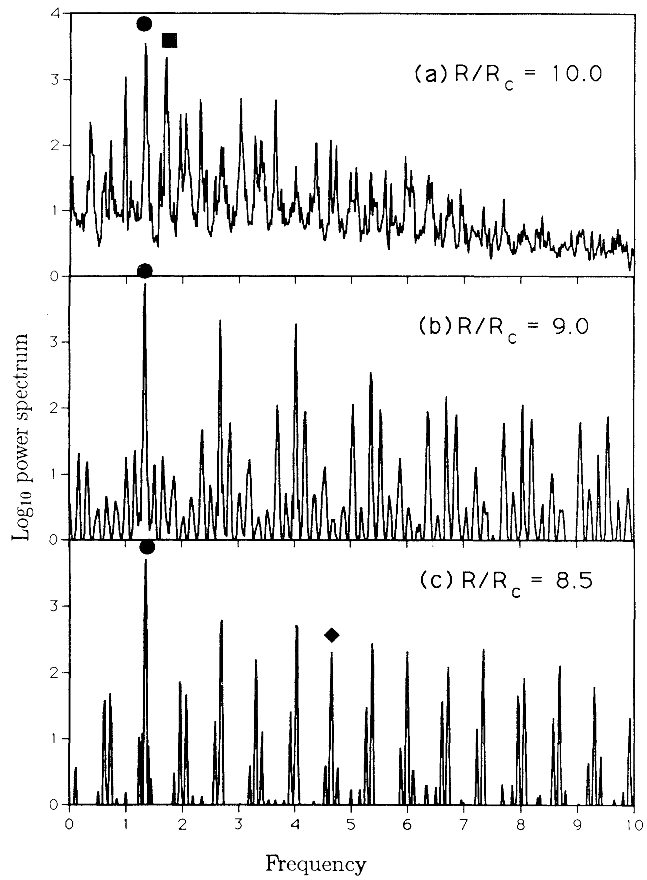

FIG. 3. Power spectra from experimental time series of the intensity of laser light scattered from the flow-visualization particles. Frequencies are normalized by the inner-cylinder frequency. ● denotes the m_1c_1 peaks; ■ denotes the m_2c_2 GS mode; ◆ denotes the frame-independent beat frequency $m_2(c_2-c_1)$. (a) GS mode, (b) intermittent flow with both ZS and GS components, and (c) ZS mode.

At $R/R_c=9.00$, the numerical values are $c_1=0.336$, and $c_2/c_1=1.29$ for the GS mode and 2.03 for the ZS mode.

Quasiperiodic modes in the Couette-Taylor system have historically been of interest because they are the state that bifurcates to chaos. Quantitative laboratory measurements have shown that the chaotic state is low dimensional, and that the transition is continuous and nonhysteretic.[12] Given that the quasiperiodic state is of the form (2), this implies that, relative to the correct rotating frame, there is a direct bifurcation from a limit cycle to a chaotic attractor. Because there is no known mathematical mechanism for this type of transition, this route to chaos is not understood. The scenario presented above provides a framework for investigating one route to chaos in this system in detail. If we assume that the unstable wavy-vortex-flow (WVF) equilibrium exists over the range of R considered here, and that the ZS and GS modes are Floquet eigenmodes of this WVF, it is likely that the aperiodic flow arises from mode competition similar to what one sees at "bicritical points,"[13] although we do not know definitely that such a point exists here.[14]

It has yet to be demonstrated quantitatively that a fully three-dimensional, time-dependent chaotic solution to the Navier-Stokes equations can be accurately represented by a few specially chosen modes. The problem of how to capture the full *spatial* structure of the flow in a low-dimensional truncation, rather than just model the temporal dynamics at a single point, is poorly understood. The chaotic flow in the regime where the GS and ZS Floquet modes are unstable is well suited to the study of this problem: The unstable rotating wave and the two complex Floquet modes represent the physically active degrees of freedom in the flow, and can be used as the first five basis functions in a spectral truncation. The ZS and GS modes are strongly coupled because they have the same spatial symmetry (i.e., m_2 is a multiple of m_1). To see this, note that both modes can be written as Fourier sums in terms of $e^{ikm_1\theta}$ for integer k; hence, their nonlinear interaction will also be of this form. If both modes have amplitude ε, then the nonlinear interaction, of order ε^2, will be a power series in $e^{im_1\theta}$ and thus act directly back on the unstable WVF. If the ZS and GS modes had different values of m_2 (say, 4 and 11), their interaction back on WVF would be of higher order in ε (ε^3). We find numerically that a truncation of five modes captures more than 90% of the energy and enstrophy in the flow. This basis set can be expanded with additional functions based on harmonics and cross products of the original five modes, a procedure which is sensitive to the choice of projection operator and requires a close examination of the physics. The convergence rate of the truncation (measured against the full numerical simulation with 32^3 modes) as a function of the number of included modes determines both how well the low-dimensional truncation resolves the spatial structure of the flow and whether the chaos is truly low dimensional. We stress that this truncation is derived from the Navier-Stokes equations (not a model), has no arbitrary constants, and incorporates adaptation of the basis functions as the Reynolds number and other parameters change. Hence, it should be valid over a wider range of physical conditions than models derived on the assumption that one is at a bicritical point.

In summary, we have used analysis and numerical simulations to describe the spatial structure and temporal behavior of two distinct modulated waves in Couette-Taylor flow. Experiments have confirmed that the modes can be present independently or can coexist in a temporally chaotic state. We have discussed how these modes, considered as the active degrees of freedom in a spectral truncation of the full equations, provide a basis for an investigation of a low-dimensional system which accurately captures both the spatial and temporal behavior of a complex three-dimensional flow. This low-dimensional truncation will be used in future work to investigate a mechanism for chaos in this flow, and tested quantitatively against full simulations.

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[a]Present address: Department of Physics, University of Colorado, Denver, CO 80309.

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[10]As is well known, the flow state is not a unique function of the external parameters.

[11]At $R/R_c=8.50$ and $R/R_c=9.00$ the uncertainty in the value of c_2/c_1 could be reduced, and at $R/R_c=9.00$ the spectral peaks could be more easily identified with the values of j and k in Eq. (2), if measurements were done in a rotating frame.

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[14]It is possible that, by tuning another parameter (the radius ratio or the axial wavelength), one could adjust the system so that the eigenvalues of both the ZS and GS modes cross the imaginary axis at the same R, and obtain a bicritical point. However, these modes can only be computed numerically, and such a parameter search would be prohibitively expensive.